Analytical Approximation of the Neutrino Oscillation Probabilities at Large θ_{13}

Tatsu Takeuchi
Virginia Tech & Kavli IPMU

February 13, 2013 @ Kavli IPMU

Work in Collaboration with

- Sanjib Agarwalla (Institute of Physics, Bhubaneswar)
- Yee Kao (Western Carolina U.)
- Naotoshi Okamura (Yamanashi U.)
- Minako Honda (Ochanomizu U.)

Based on:

- hep-ph/0602115
- hep-ph/0603268
- arXiv:0707.4545 [hep-ph]

Matter Effect on Neutrino Oscillation

- The neutrino beams of long-baseline neutrino oscillation experiments performed on the Earth necessarily traverse Earth's interior
  $\rightarrow$ Matter effects must be taken into account.

- Exact analytical expressions exist for the oscillation probabilities for constant density matter, but are too complicated to be of practical use.

- Numerical calculations obscure the physics.

- Simple analytical approximations to the oscillation probabilities in matter would go a long way in helping us analyze the potential of various experiments.

Neutrino Mixing Matrix

$$\text{flavor eigenstates}: \quad |\nu_\alpha\rangle \quad (\alpha = e, \mu, \tau)$$

$$\text{mass eigenstates}: \quad |\nu_j\rangle \quad (j = 1, 2, 3)$$

$$\text{PMNS matrix}: \quad (V_{PMNS})_{\alpha j} = \langle \nu_\alpha | \nu_j \rangle$$

$$V_{PMNS} = UP$$

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{i\alpha_{21}/2} & 0 \\ 0 & 0 & e^{i\alpha_{31}/2} \end{bmatrix}$$

Neutrino Oscillation in Vacuum

$$A_{\beta\alpha} = \left\langle \nu_\beta \middle| \nu_{\alpha,0}(x = L) \right\rangle = \left[\exp\left(-i \frac{H_0}{2E} L \right) \right]_{\beta\alpha}, \qquad H_0 = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger,$$

$$P(\nu_\alpha \to \nu_\beta)$$

$$= \left| A_{\beta\alpha} \right|^2$$

$$= \delta_{\alpha\beta} - 4 \sum_{j>k} \Re\left(U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \right) \sin^2 \frac{\Delta_{jk}}{2} + 2 \sum_{j>k} \Im\left(U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \right) \sin \Delta_{jk},$$

$$\Delta_{jk} = \frac{\delta m_{jk}^2}{2E} L$$

Neutrino Oscillation in Matter

$$H_a = U\begin{bmatrix} 1 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \tilde{U}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tilde{U}^\dagger,$$

$$P(\nu_\alpha \to \nu_\beta)$$

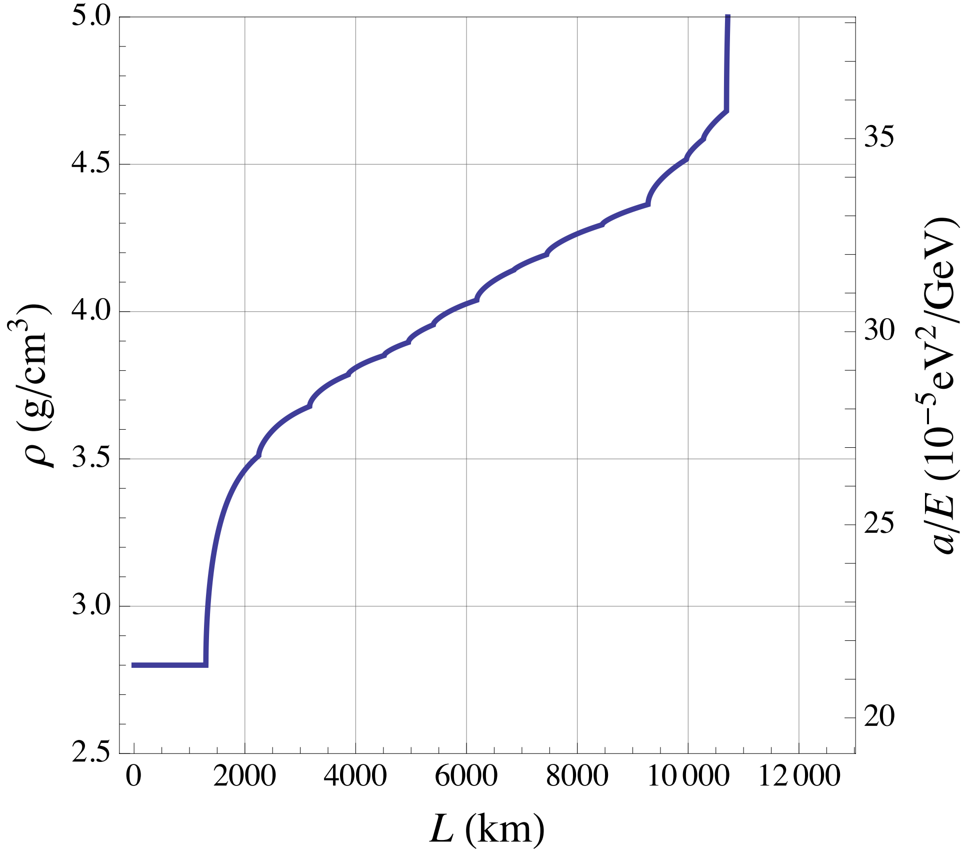
$$= \delta_{\alpha\beta} - 4\sum_{j>k} \Re\left(\tilde{U}^*_{\alpha j}\tilde{U}_{\beta j}\tilde{U}_{\alpha k}\tilde{U}^*_{\beta k}\right)\sin^2\frac{\tilde{\Delta}_{jk}}{2} + 2\sum_{j>k} \Im\left(\tilde{U}^*_{\alpha j}\tilde{U}_{\beta j}\tilde{U}_{\alpha k}\tilde{U}^*_{\beta k}\right)\sin\tilde{\Delta}_{jk},$$

$$\tilde{\Delta}_{jk} = \frac{\delta\lambda_{jk}}{2E}L, \qquad \delta\lambda_{jk} = \lambda_j - \lambda_k, \qquad a = 2\sqrt{2}G_F N_e E.$$

Can the probabilities be expressed as simple functions of a ?

Matter Effect Parameter a

$$a = 2\sqrt{2} G_F N_e E \quad = 7.63 \times 10^{-5} (\mathrm{eV}^2) \left(\frac{\rho}{\mathrm{g/cm^3}} \right) \left(\frac{E}{\mathrm{GeV}} \right)$$



Example: Formula of Cervera et al.

- A. Cervera, A. Donini, M. B. Gavela, J. J. Gomez Cádenas, P. Hernández, O. Mena, and S. Rigolin, Nuclear Physics B **579** (2000) 17-55.

$$P(\nu_\mu \to \nu_e) \approx \sin^2 2\theta_{13} \sin^2 2\theta_{23} \frac{\sin^2\left[(1-A)\Delta\right]}{(1-A)^2}$$

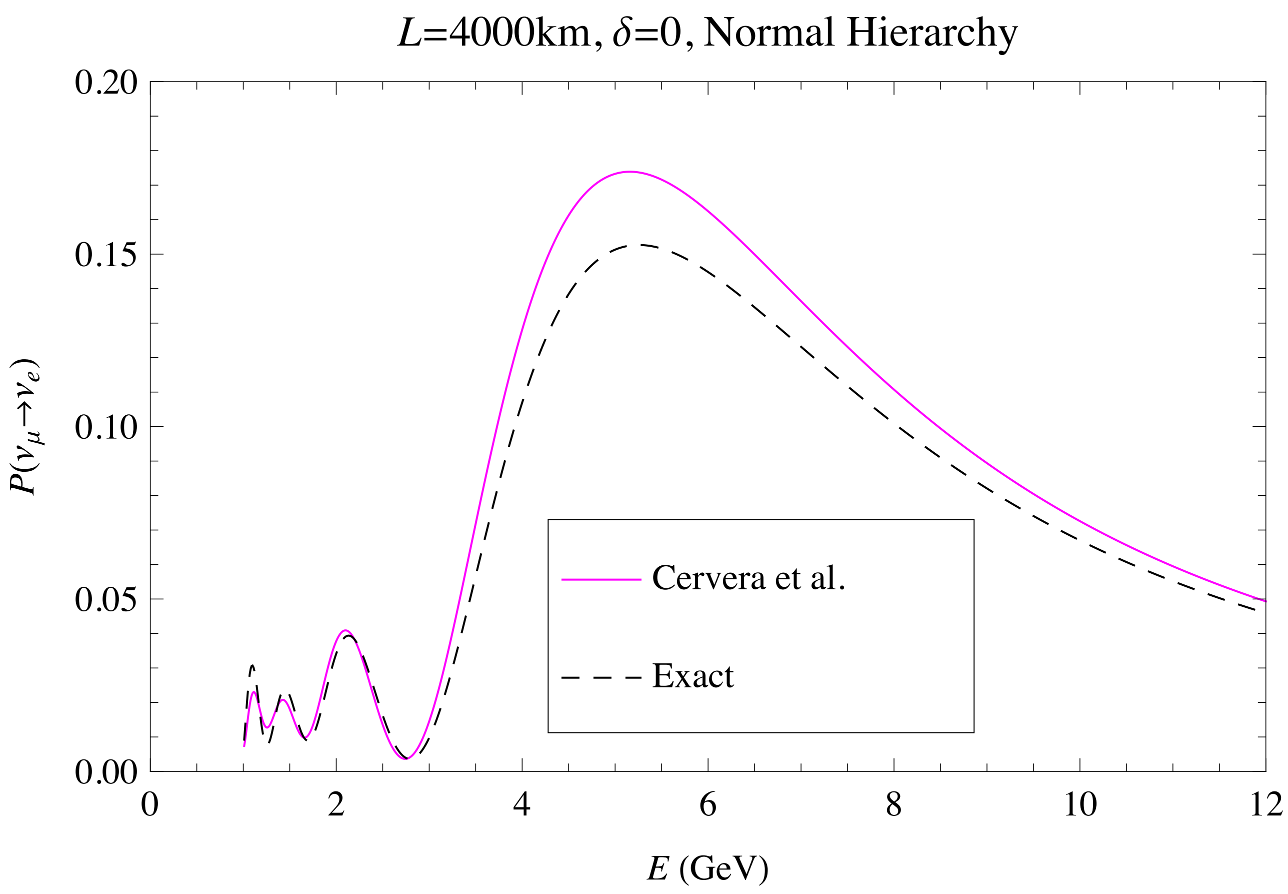
$$-\alpha \sin 2\theta_{13} \cos\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin\delta \sin\Delta \frac{\sin\left(A\Delta\right)}{A} \frac{\sin\left[(1-A)\Delta\right]}{(1-A)}$$

$$+\alpha \sin 2\theta_{13} \cos\theta_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos\delta \cos\Delta \frac{\sin\left(A\Delta\right)}{A} \frac{\sin\left[(1-A)\Delta\right]}{(1-A)}$$

$$+\alpha^2 \cos^2\theta_{23} \sin^2 2\theta_{12} \frac{\sin^2\left(A\Delta\right)}{A^2},$$

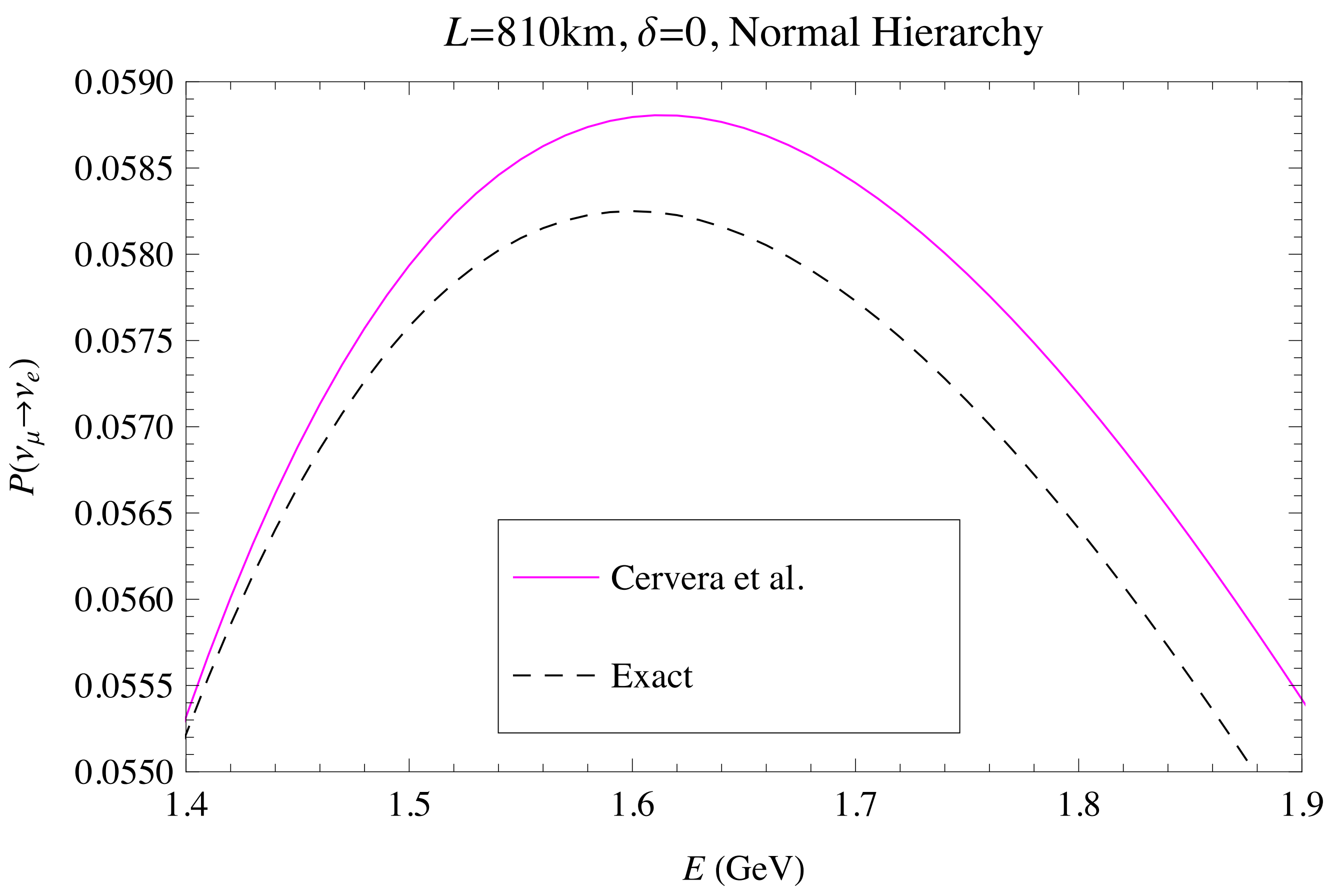
$$\alpha = \frac{\delta m_{21}^2}{\delta m_{31}^2} \approx 0.03, \quad A = \frac{a}{\delta m_{31}^2}, \quad \Delta = \frac{\delta m_{31}^2}{4E} L, \quad \sin 2\theta_{13} \approx 0.3$$

L=4000 km

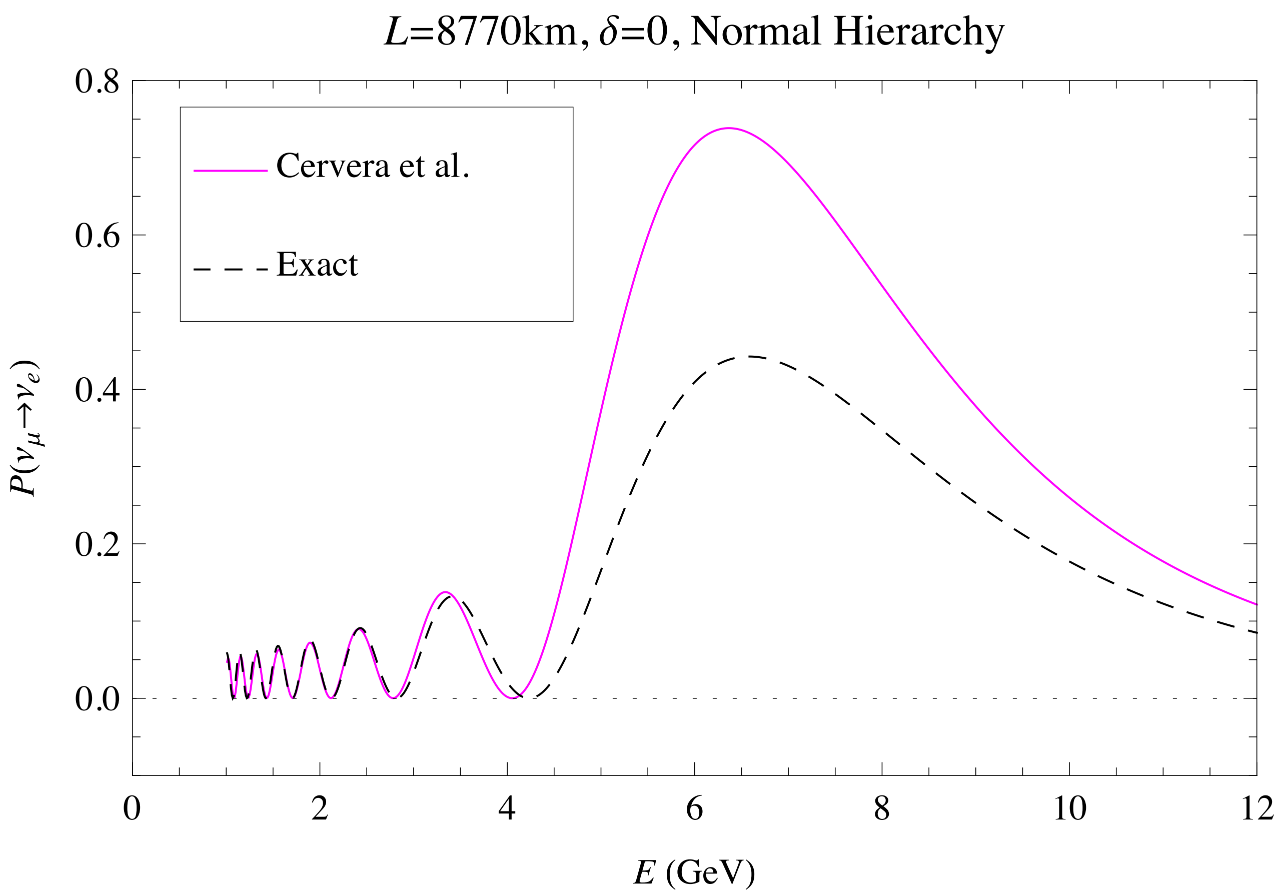


L=4000km, δ=0, Normal Hierarchy
0.20
0.15
0.10
0.05
0.00
$P(\nu_\mu \to \nu_e)$
0
2
4
6
8
10
12
E (GeV)
Cervera et al.
Exact

L=810 km (Fermilab→NOvA)



L=8770 km (CERN→HyperK)



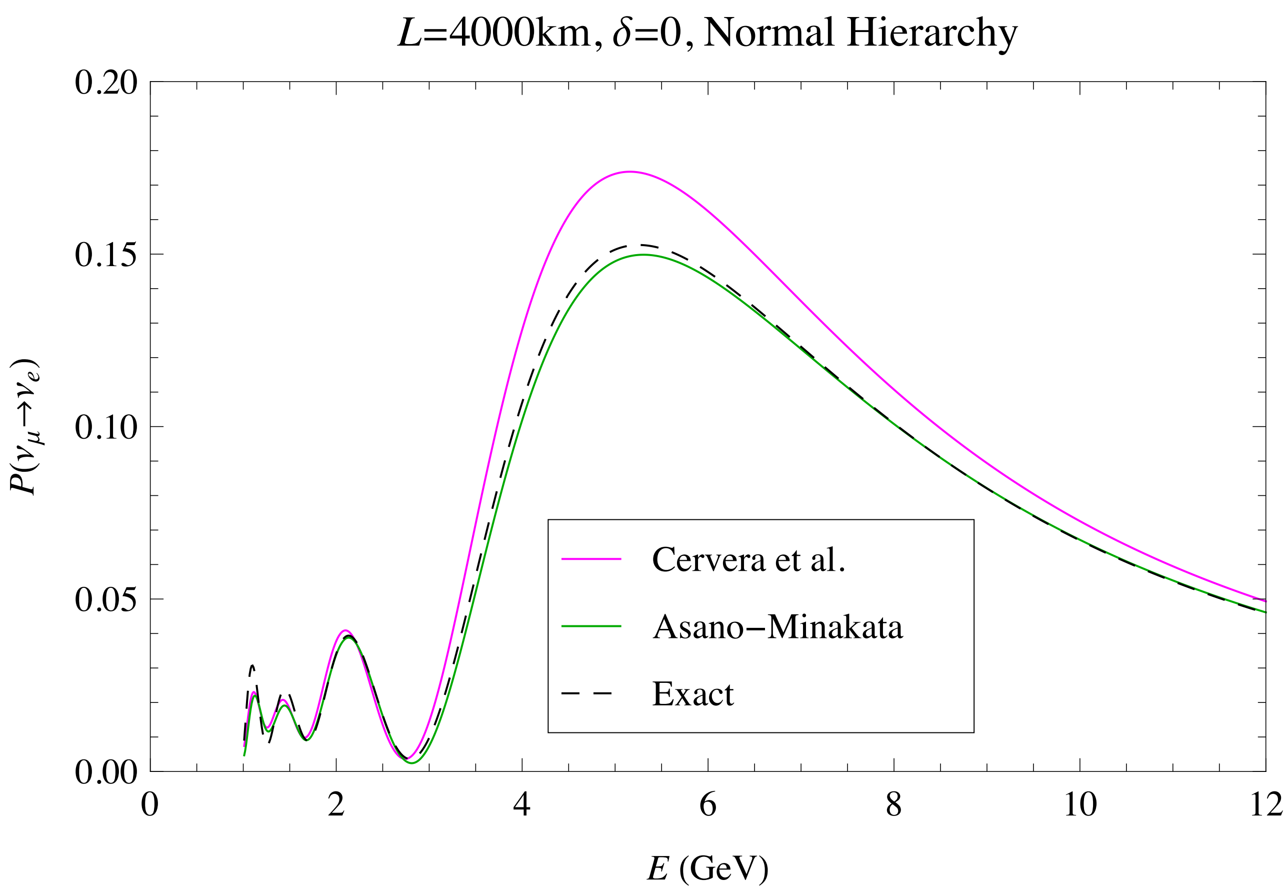
Asano-Minakata

- K. Asano and H. Minakata, JHEP 1106 (2011) 022 [arXiv:1103.4387[hep-ph]]

$$P_{AM}(\nu_\mu \to \nu_e) \approx P_C(\nu_\mu \to \nu_e)$$
$$-4\sin^2\theta_{23}\left[\sin^4\theta_{13}\left(\frac{1+A}{1-A}\right)^2 - 2\alpha\sin^2\theta_{13}\sin^2\theta_{12}\left(\frac{A}{1-A}\right)\right]\frac{\sin^2[(1-A)\Delta]}{(1-A)^2}$$
$$+4\sin^2\theta_{23}\left[2\sin^4\theta_{13}\left(\frac{A}{1-A}\right)-\alpha\sin^2\theta_{13}\sin^2\theta_{12}\right]\frac{\Delta\sin[2(1-A)\Delta]}{(1-A)^2},$$

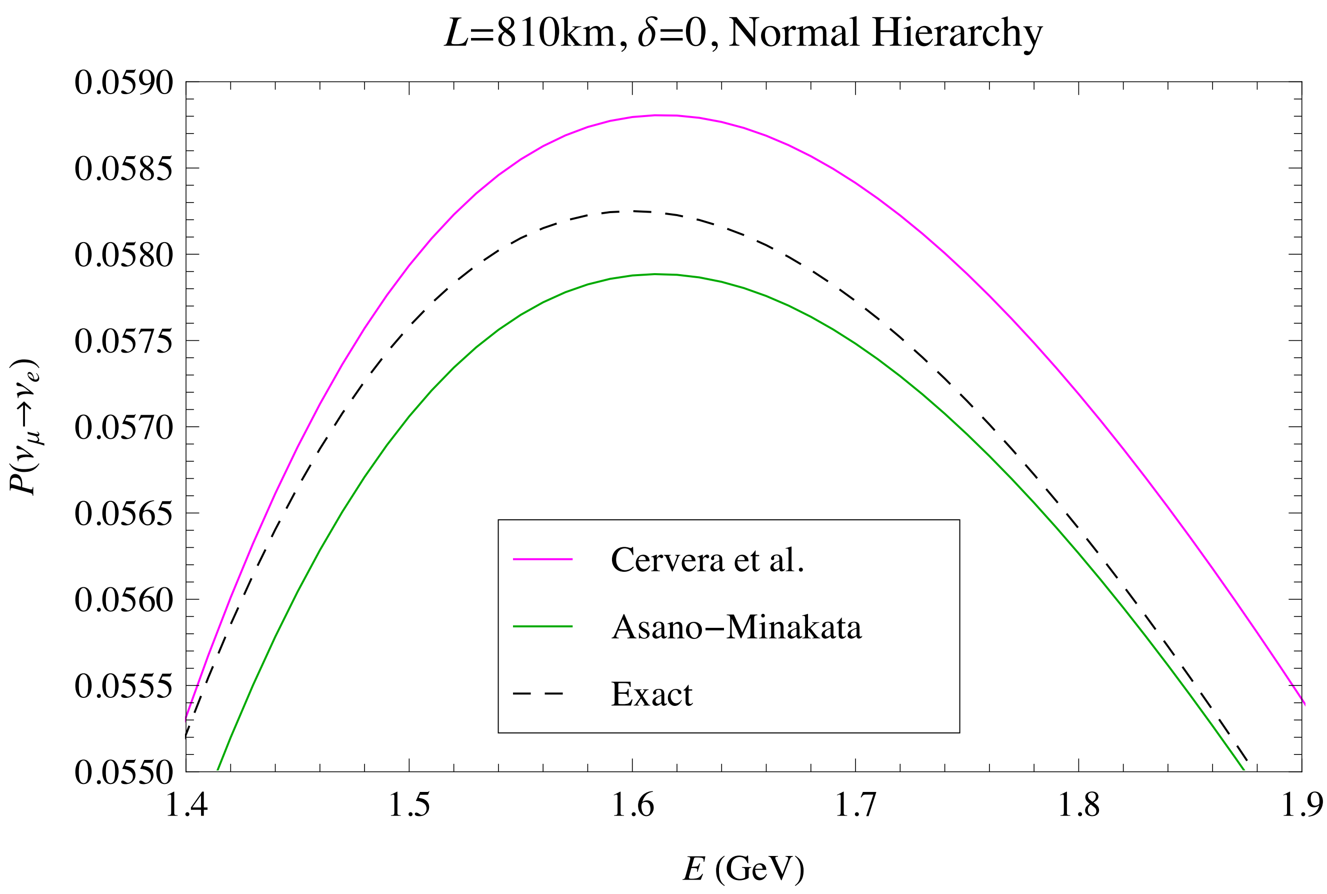
$$\sin^4\theta_{13} \approx 0.0005, \qquad \alpha\sin^2\theta_{13} \approx 0.0007$$

L=4000 km

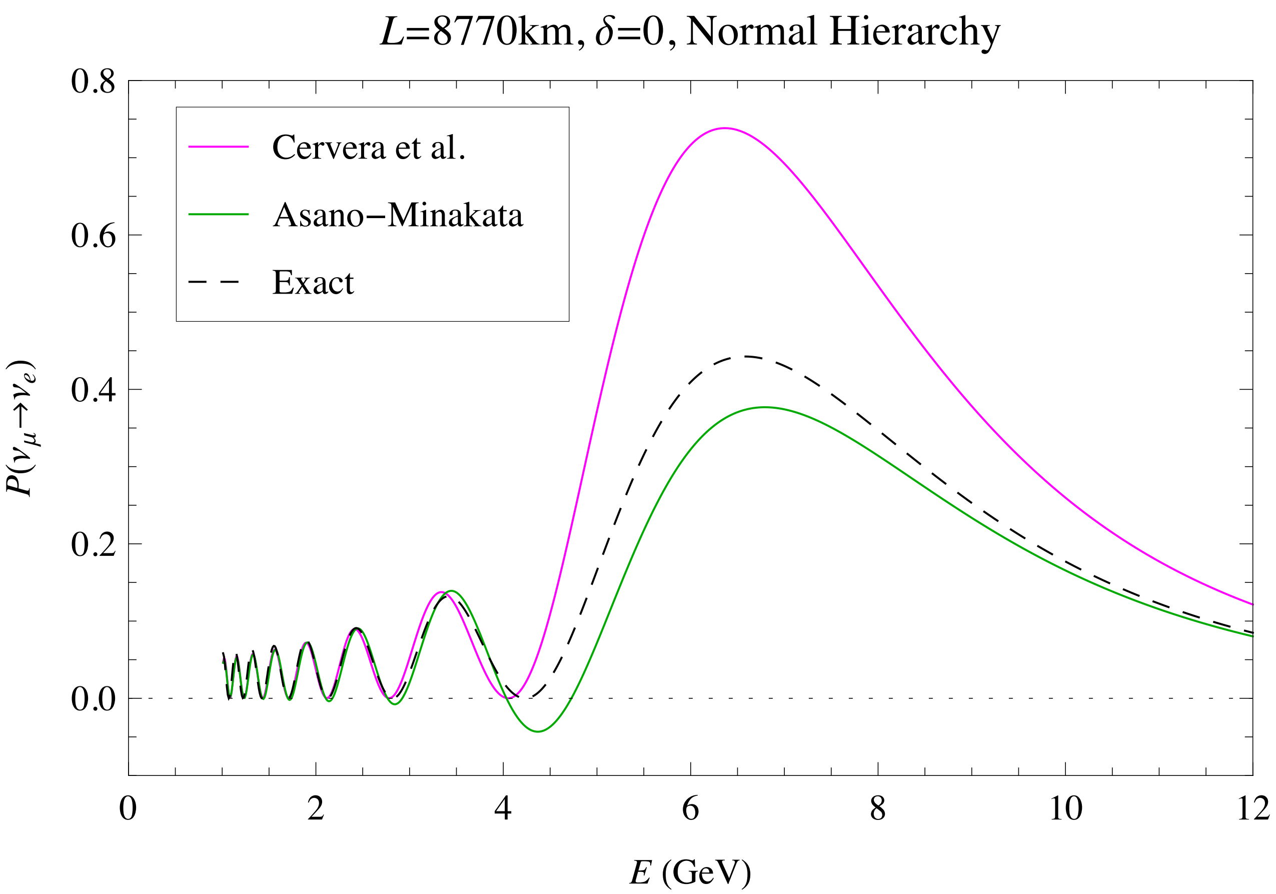


L=4000km, δ=0, Normal Hierarchy
0.20
0.15
0.10
0.05
0.00
$P(\nu_\mu \to \nu_e)$
0
2
4
6
8
10
12
E (GeV)
Cervera et al.
Asano−Minakata
Exact

L=810 km (Fermilab→NOvA)



L=8770 km (CERN→HyperK)



Our Approach

- Use the expressions for the vacuum oscillation probabilities as is, but make the following replacements:

$$\theta_{12} \to \theta'_{12}, \quad \theta_{13} \to \theta'_{13}, \quad \delta m^2_{jk} \to \lambda_j - \lambda_k$$

where

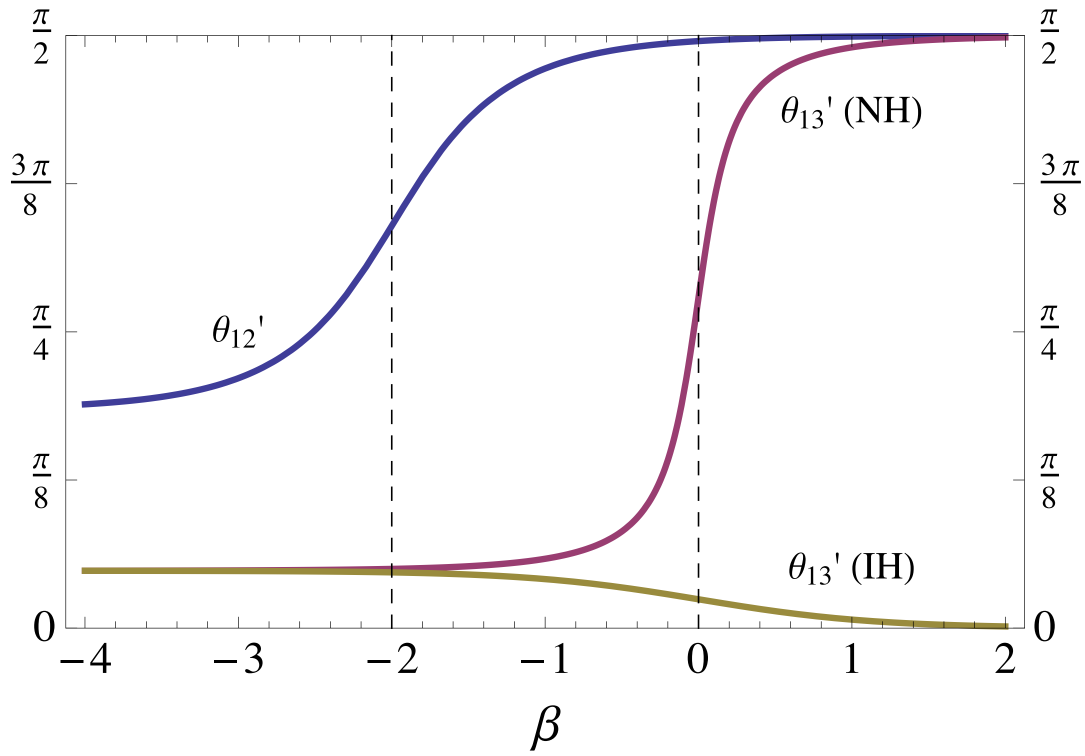
$$\tan 2\theta'_{12} = \frac{(\delta m^2_{21}/c^2_{13})\sin 2\theta_{12}}{(\delta m^2_{21}/c^2_{13})\cos 2\theta_{12} - a}, \qquad \tan 2\theta'_{13} = \frac{(\delta m^2_{31} - \delta m^2_{21} s^2_{12})\sin 2\theta_{13}}{(\delta m^2_{31} - \delta m^2_{21} s^2_{12})\cos 2\theta_{13} - a},$$

$$\lambda_1 = \lambda'_-$$
$$\lambda_2 = \lambda''_\mp$$
$$\lambda_3 = \lambda''_\pm$$

$$\lambda'_\pm = \frac{(\delta m^2_{21} + a c^2_{13}) \pm \sqrt{(\delta m^2_{21} - a c^2_{13})^2 + 4 a c^2_{13} s^2_{12} \delta m^2_{21}}}{2}$$

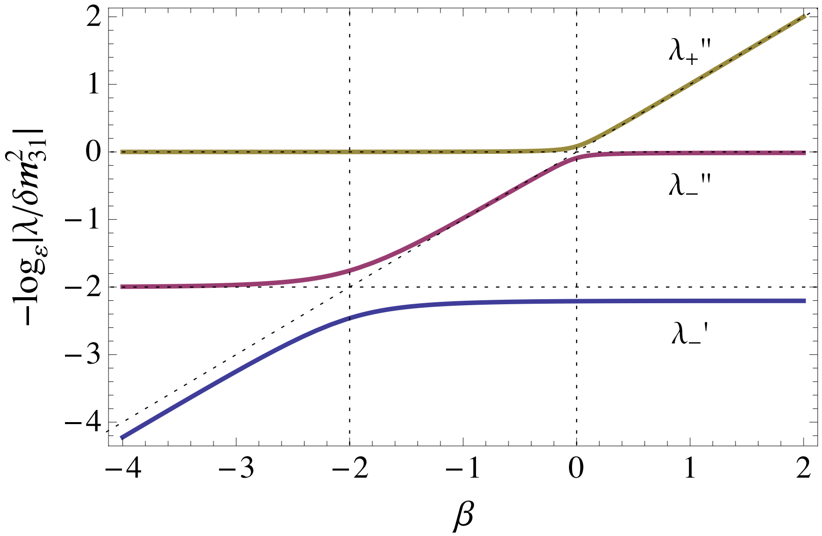
$$\lambda''_\pm = \frac{\left[\lambda'_+ + (\delta m^2_{31} + a s^2_{13})\right] \pm \sqrt{\left[\lambda'_+ - (\delta m^2_{31} + a s^2_{13})\right]^2 + 4 a^2 s'^2_{12} c^2_{13} s^2_{13}}}{2}$$

a-dependence of effective mixing angles

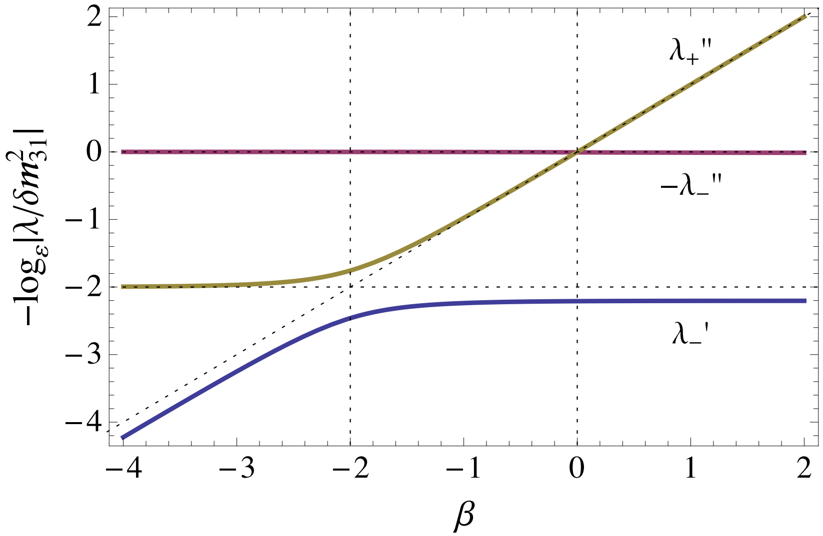


$$\frac{a}{|\delta m_{31}^2|} = \varepsilon^{-\beta}, \qquad \varepsilon = \sqrt{\frac{\delta m_{21}^2}{|\delta m_{31}^2|}} \approx 0.17$$

α-dependence of effective mass-squares

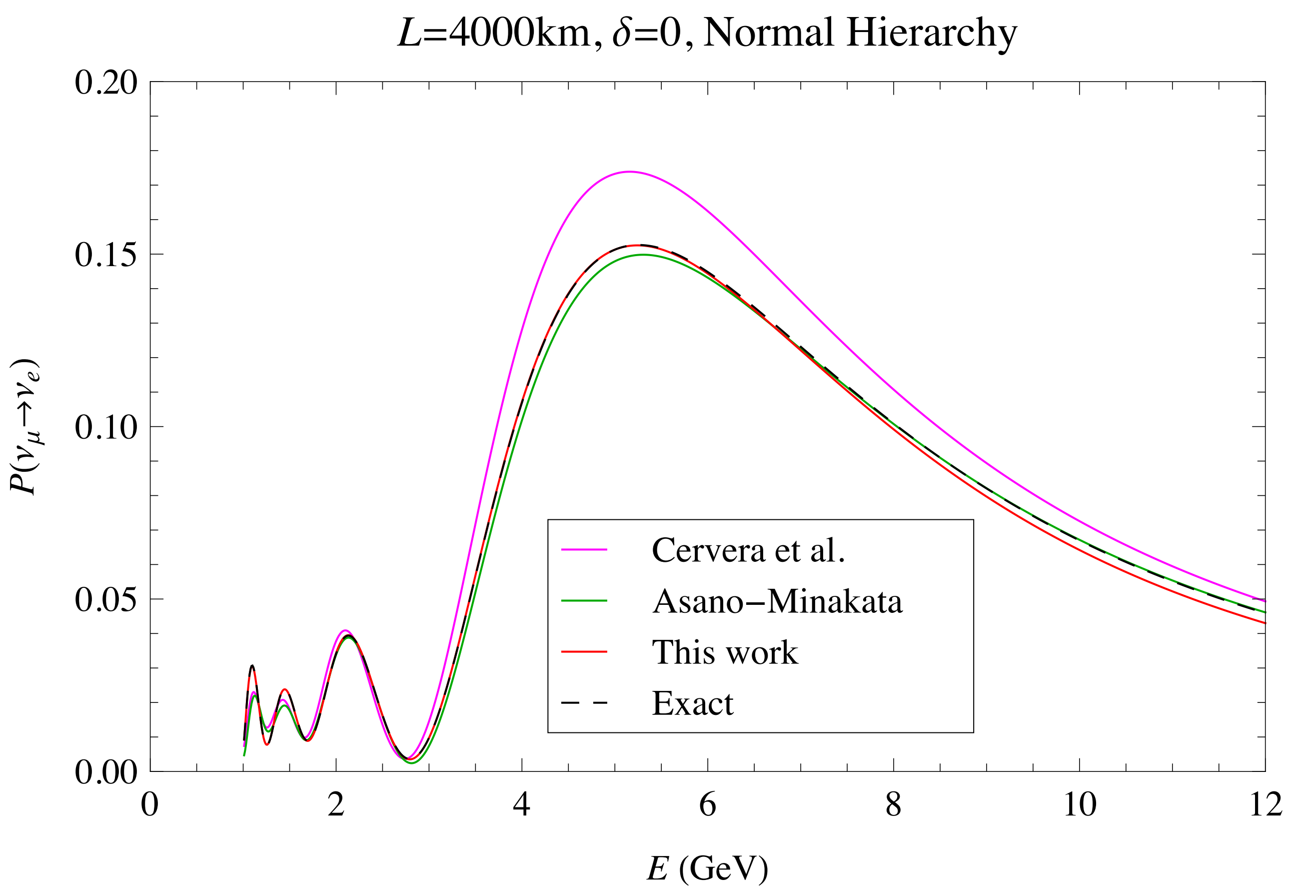


Normal Hierarchy



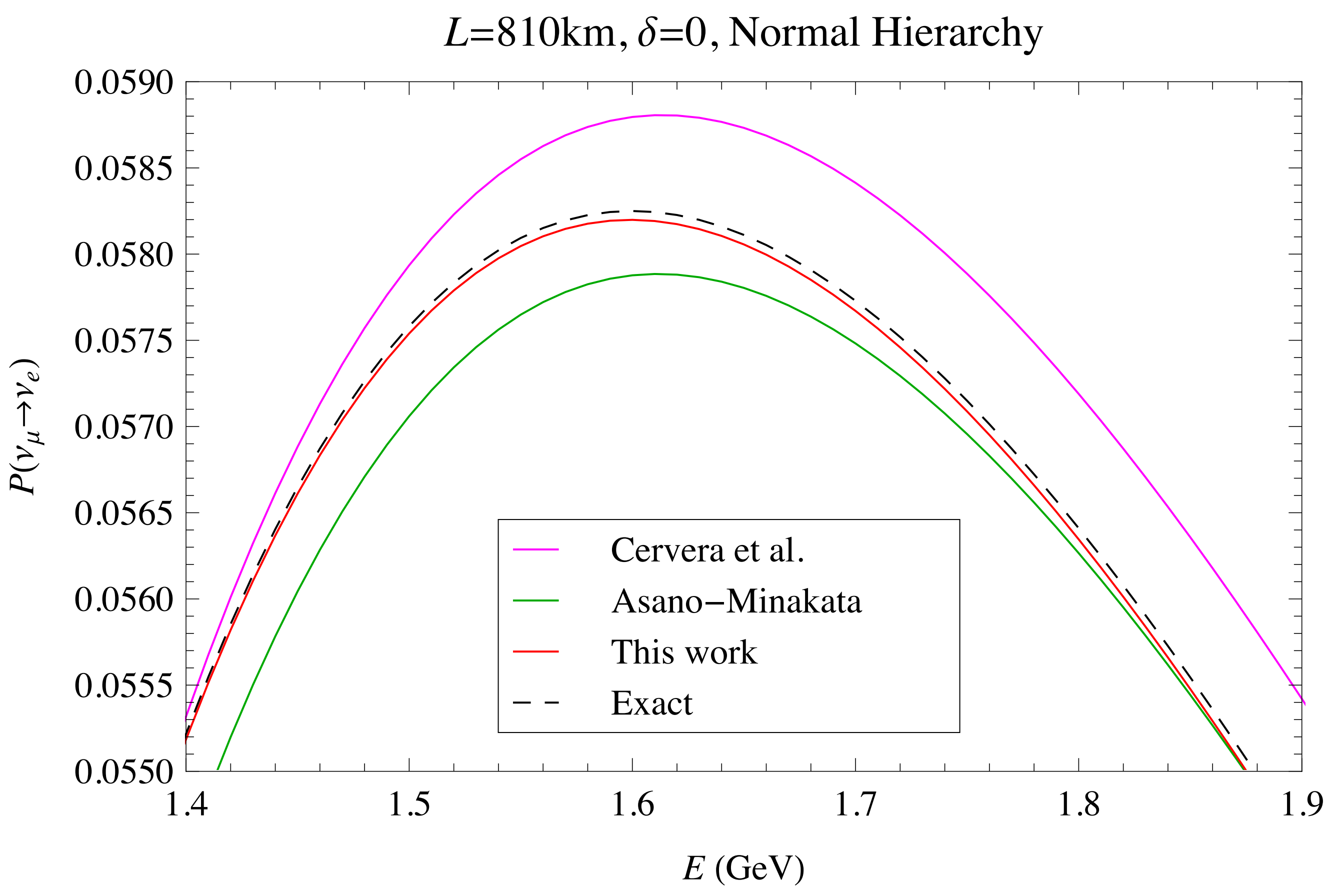
Inverted Hierarchy

L=4000 km

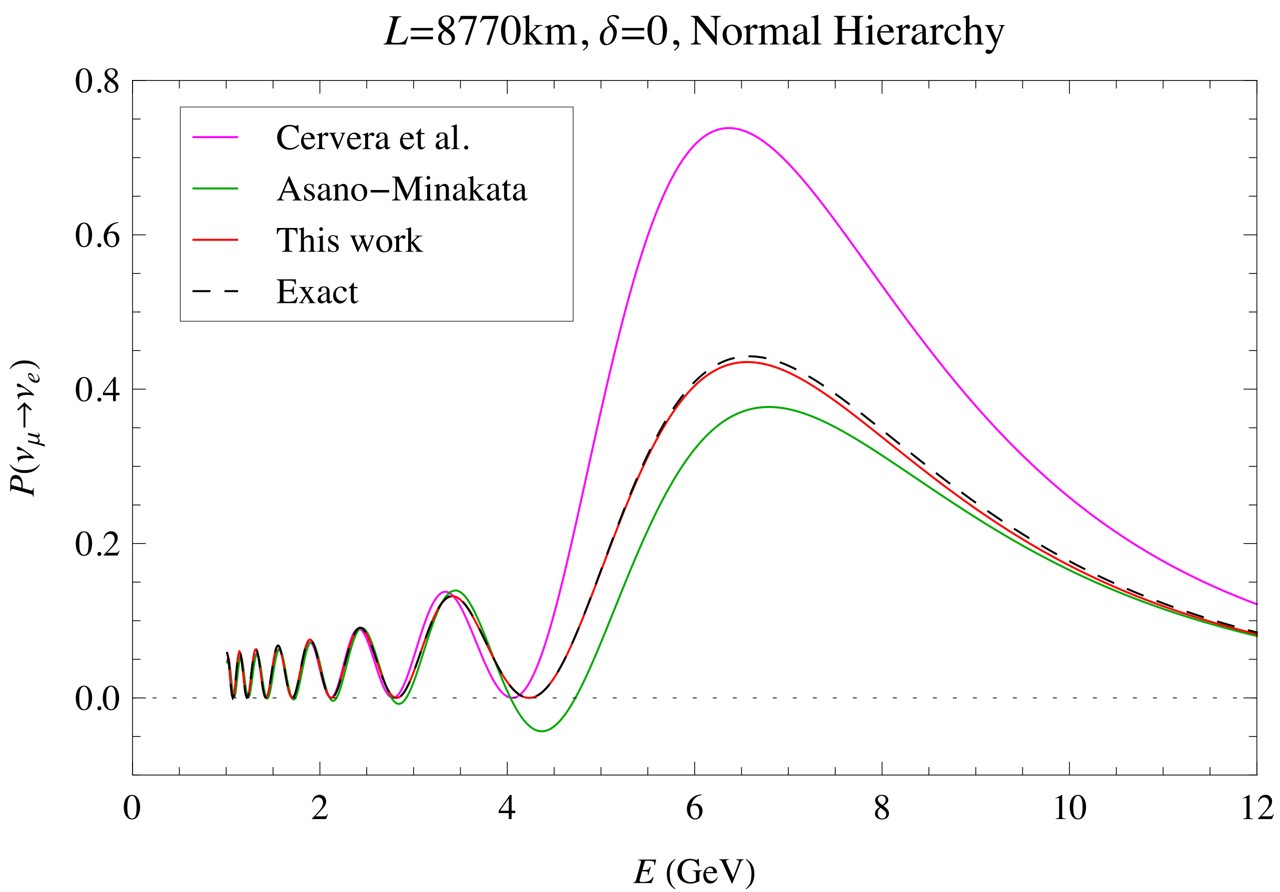


L=4000km, δ=0, Normal Hierarchy
0.20
0.15
0.10
0.05
0.00
$P(\nu_\mu \to \nu_e)$
0
2
4
6
8
10
12
E (GeV)
Cervera et al.
Asano−Minakata
This work
Exact

L=810 km (Fermilab→NOvA)



L=8770 km (CERN→HyperK)



Diagonalization of the Effective Hamiltonian

$$H_a = U \begin{bmatrix} 1 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \tilde{U} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tilde{U}^\dagger,$$

$$H_a' = Q^\dagger U^\dagger H_a U Q$$

$$= Q^\dagger \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} + U^\dagger \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} U \right\} Q$$

$$= \begin{bmatrix} a c_{12}^2 c_{13}^2 & a c_{12} s_{12} c_{13}^2 & a c_{12} c_{13} s_{13} \\ a c_{12} s_{12} c_{13}^2 & a s_{12}^2 c_{13}^2 + \delta m_{21}^2 & a s_{12} c_{13} s_{13} \\ a c_{12} c_{13} s_{13} & a s_{12} c_{13} s_{13} & a s_{13}^2 + \delta m_{31}^2 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\delta} \end{bmatrix}.$$

Jacobi Method (1846)

- Carl Gustav Jacob Jacobi (1804-1851)
- “Über ein leichtes Verfahren die in der Theorie der Säcularstörungen vorkommenden Gleichungen numerisch aufzulösen,”
  Crelle’s Journal **30** (1846) 51-94.

1st Rotation

$$H'_a = \begin{bmatrix} ac_{12}^2c_{13}^2 & ac_{12}s_{12}c_{13}^2 & ac_{12}c_{13}s_{13} \\ ac_{12}s_{12}c_{13}^2 & as_{12}^2c_{13}^2 + \delta m_{21}^2 & as_{12}c_{13}s_{13} \\ ac_{12}c_{13}s_{13} & as_{12}c_{13}s_{13} & as_{13}^2 + \delta m_{31}^2 \end{bmatrix}$$

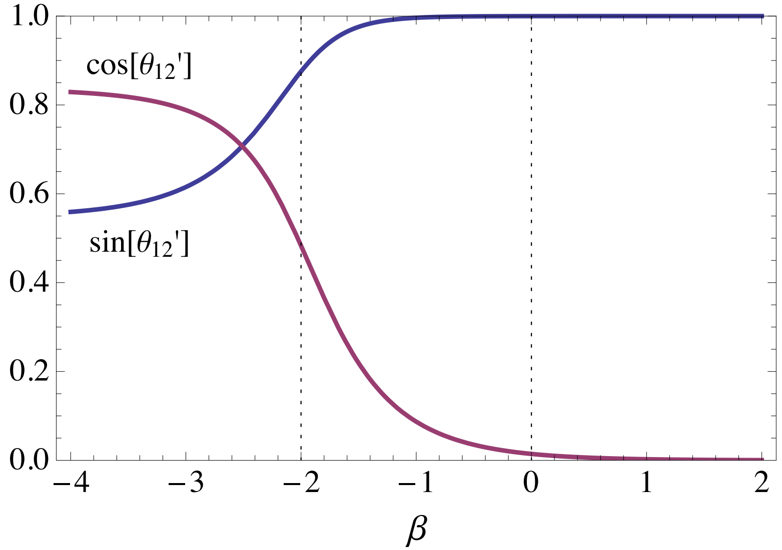
$$V = \begin{bmatrix} c_\varphi & s_\varphi & 0 \\ -s_\varphi & c_\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \tan 2\varphi = \frac{a\sin 2\theta_{12}}{(\delta m_{21}^2 / c_{13}^2) - a\cos 2\theta_{12}}$$

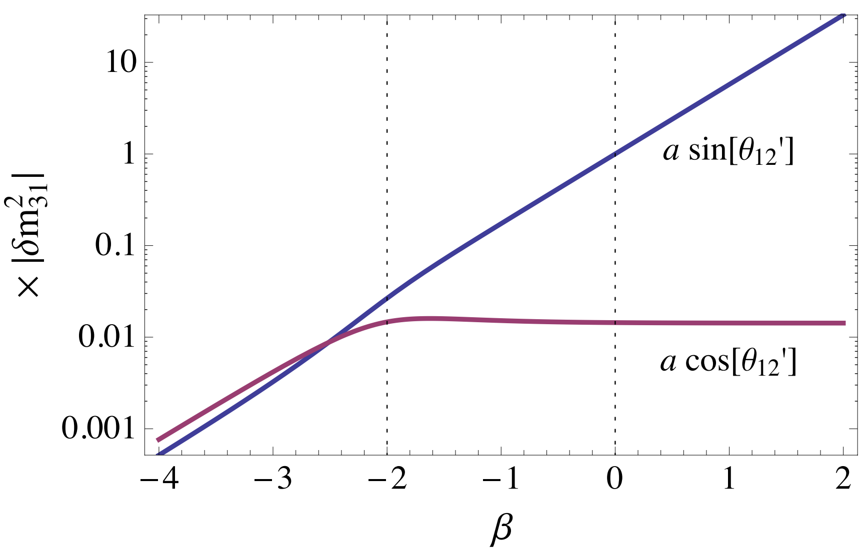
$$H''_a = V^\dagger H'_a V = \begin{bmatrix} \lambda'_- & 0 & ac'_{12}c_{13}s_{13} \\ 0 & \lambda'_+ & as'_{12}c_{13}s_{13} \\ ac'_{12}c_{13}s_{13} & as'_{12}c_{13}s_{13} & as_{13}^2 + \delta m_{31}^2 \end{bmatrix},$$

$$\theta'_{12} = \theta_{12} + \varphi, \qquad \lambda'_\pm = \frac{(\delta m_{21}^2 + ac_{13}^2) \pm \sqrt{(\delta m_{21}^2 - ac_{13}^2)^2 + 4ac_{13}^2 s_{12}^2 \delta m_{21}^2}}{2}$$

1st Rotation

$$\tan 2\theta'_{12} = \tan 2(\theta_{12}+\varphi) = \frac{(\delta m^2_{21}/c^2_{13})\sin 2\theta_{12}}{(\delta m^2_{21}/c^2_{13})\cos 2\theta_{12} - a}$$





2nd Rotation

$$H_a'' = \begin{bmatrix} \lambda_-' & 0 & ac_{12}'c_{13}s_{13} \\ 0 & \lambda_+' & as_{12}'c_{13}s_{13} \\ ac_{12}'c_{13}s_{13} & as_{12}'c_{13}s_{13} & as_{13}^2 + \delta m_{31}^2 \end{bmatrix}$$

$$W = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & s_\phi \\ 0 & -s_\phi & c_\phi \end{bmatrix}, \qquad \tan 2\phi = \frac{as_{12}' \sin 2\theta_{13}}{\delta m_{31}^2 + as_{13}^2 - \lambda_+'} \approx \frac{a \sin 2\theta_{13}}{(\delta m_{31}^2 - \delta m_{31}^2 s_{12}^2) - a\cos 2\theta_{13}}$$

$$H_a''' = W^\dagger H_a'' W = \begin{bmatrix} \lambda_-' & -ac_{12}'c_{13}s_{13}s_\phi & ac_{12}'c_{13}s_{13}c_\phi \\ -ac_{12}'c_{13}s_{13}s_\phi & \lambda_\mp'' & 0 \\ ac_{12}'c_{13}s_{13}c_\phi & 0 & \lambda_\pm'' \end{bmatrix},$$

$$\lambda_\pm'' = \frac{\left[\lambda_+' + (\delta m_{31}^2 + as_{13}^2)\right] \pm \sqrt{\left[\lambda_+' - (\delta m_{31}^2 + as_{13}^2)\right]^2 + 4a^2 s_{12}'^2 c_{13}^2 s_{13}^2}}{2}$$

Effective Mixing Matrix

$$
\begin{aligned}
\tilde{U} &= UQVW \\
&= R_{23}(\theta_{23},0)R_{13}(\theta_{13},\delta)R_{12}(\theta_{12},0)\,Q\,R_{12}(\varphi,0)R_{23}(\phi,0) \\
&= R_{23}(\theta_{23},0)\,Q\,R_{13}(\theta_{13},0)R_{12}(\theta_{12},0)R_{12}(\varphi,0)R_{23}(\phi,0) \\
&= R_{23}(\theta_{23},0)\,Q\,R_{13}(\theta_{13},0)R_{12}(\theta_{12}+\varphi,0)R_{23}(\phi,0) \\
&= R_{23}(\theta_{23},0)\,Q\,R_{13}(\theta_{13},0)R_{12}(\theta'_{12},0)R_{23}(\phi,0) \\
&\approx R_{23}(\theta_{23},0)\,Q\,R_{13}(\theta_{13},0)R_{13}(\phi,0)R_{12}(\theta'_{12},0) \\
&= R_{23}(\theta_{23},0)\,Q\,R_{13}(\theta_{13}+\phi,0)R_{12}(\theta'_{12},0) \\
&= R_{23}(\theta_{23},0)\,Q\,R_{13}(\theta'_{13},0)R_{12}(\theta'_{12},0) \\
&= R_{23}(\theta_{23},0)R_{13}(\theta'_{13},\delta)R_{12}(\theta'_{12},0)\,Q
\end{aligned}
$$

$$\theta'_{13} = \theta_{13} + \phi$$

Effective Mixing Matrix

$$R_{12}(\theta'_{12},0)R_{23}(\phi,0)$$

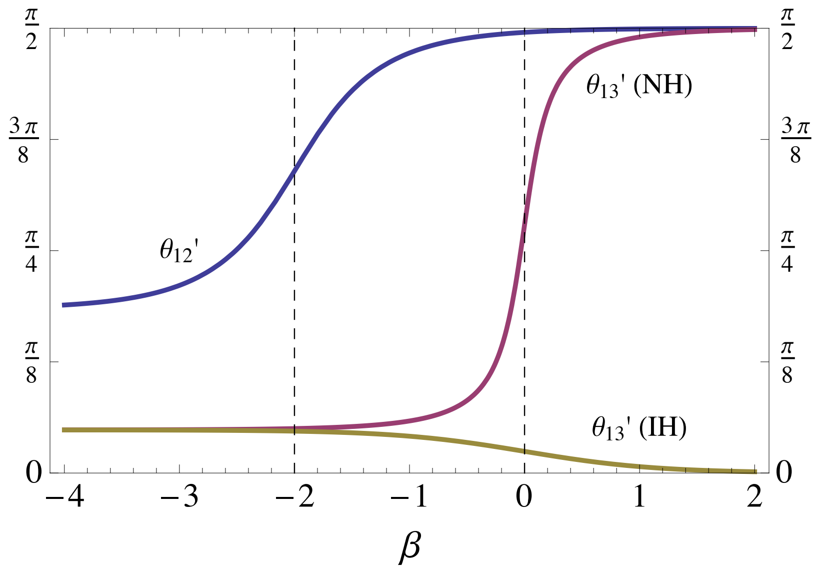
$$=\begin{bmatrix} c'_{12} & s'_{12} & 0 \\ -s'_{12} & c'_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & s_\phi \\ 0 & -s_\phi & c_\phi \end{bmatrix} \approx \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\phi & s_\phi \\ 0 & -s_\phi & c_\phi \end{bmatrix}$$

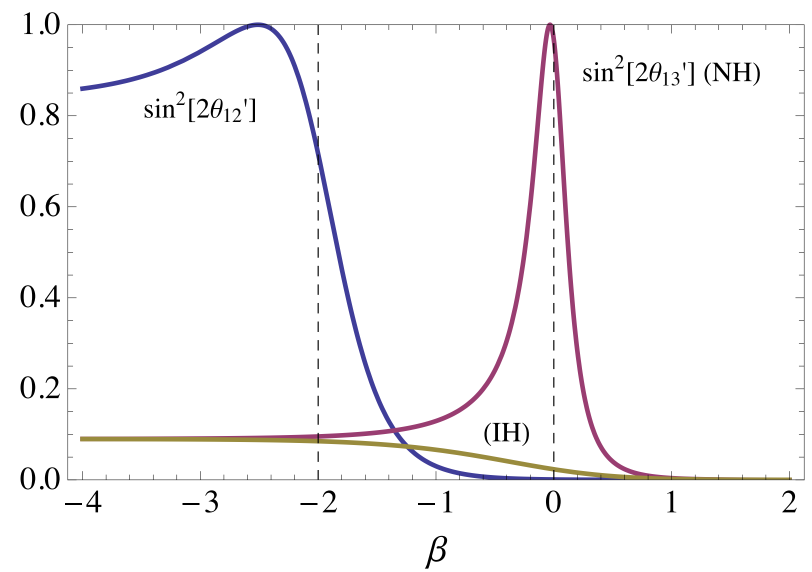
$$=\begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix}\begin{bmatrix} c'_{12} & s'_{12} & 0 \\ -s'_{12} & c'_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= R_{13}(\phi,0)R_{12}(\theta'_{12},0)$$

Effective Mixing Angles

$$\tan 2\theta'_{12} = \frac{(\delta m^2_{21}/c^2_{13})\sin 2\theta_{12}}{(\delta m^2_{21}/c^2_{13})\cos 2\theta_{12} - a}, \qquad \tan 2\theta'_{13} = \frac{(\delta m^2_{31} - \delta m^2_{21}s^2_{12})\sin 2\theta_{13}}{(\delta m^2_{31} - \delta m^2_{21}s^2_{12})\cos 2\theta_{13} - a},$$





Application 1: Mass Hierarchy Dependence

$$P(\nu_e \to \nu_e)$$

$$= 1 - 4\left|\tilde{U}_{e2}\right|^2 \left(1 - \left|\tilde{U}_{e2}\right|^2\right)\sin^2\frac{\tilde{\Delta}_{21}}{2} - 4\left|\tilde{U}_{e3}\right|^2 \left(1 - \left|\tilde{U}_{e3}\right|^2\right)\sin^2\frac{\tilde{\Delta}_{31}}{2}$$

$$+ 2\left|\tilde{U}_{e2}\right|^2 \left|\tilde{U}_{e3}\right|^2 \left(4\sin^2\frac{\tilde{\Delta}_{21}}{2}\sin^2\frac{\tilde{\Delta}_{31}}{2} + \sin\tilde{\Delta}_{21}\sin\tilde{\Delta}_{31}\right)$$

$$= 1 - 4c_{13}'^2 s_{12}'^2 \left(1 - c_{13}'^2 s_{12}'^2\right)\sin^2\frac{\tilde{\Delta}_{21}}{2} - \sin^2\left(2\theta_{13}'\right)\sin^2\frac{\tilde{\Delta}_{31}}{2}$$

$$+ s_{12}'^2 \sin^2\left(2\theta_{13}'\right)\left(2\sin^2\frac{\tilde{\Delta}_{21}}{2}\sin^2\frac{\tilde{\Delta}_{31}}{2} + \frac{1}{2}\sin\tilde{\Delta}_{21}\sin\tilde{\Delta}_{31}\right)$$

$$\xrightarrow{s_{12}' \approx 1} 1 - \sin^2\left(2\theta_{13}'\right)\left(\sin^2\frac{\tilde{\Delta}_{21}}{2} + \sin^2\frac{\tilde{\Delta}_{31}}{2} - 2\sin^2\frac{\tilde{\Delta}_{21}}{2}\sin^2\frac{\tilde{\Delta}_{31}}{2} - \frac{1}{2}\sin\tilde{\Delta}_{21}\sin\tilde{\Delta}_{31}\right)$$

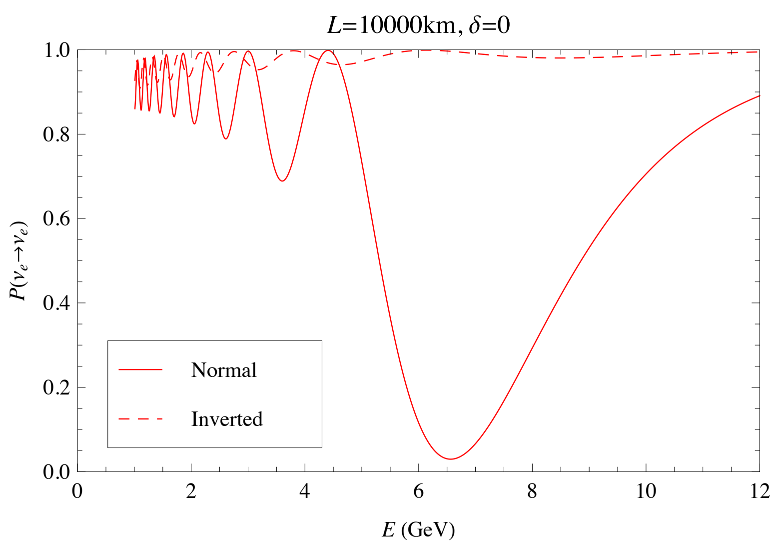
$$= 1 - \sin^2\left(2\theta_{13}'\right)\sin^2\frac{\tilde{\Delta}_{32}}{2}$$

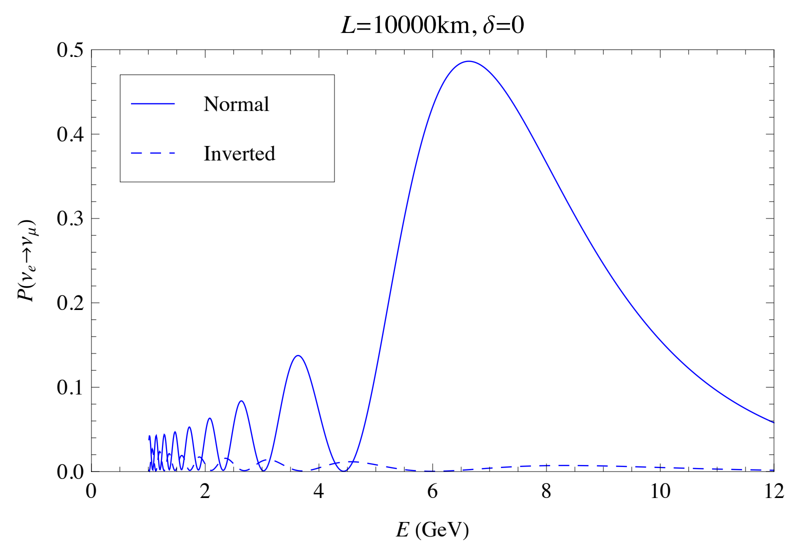
Application 1: Mass Hierarchy Dependence

Demand the oscillation term is maximized for normal hierarchy:

$$2\theta'_{13} \approx \frac{\pi}{2} \quad \rightarrow \quad a \approx \delta m_{31}^2, \qquad \frac{\tilde{\Delta}_{32}}{2} \approx \frac{s_{13}\delta m_{31}^2}{2E} L \approx \frac{\pi}{2}$$

This is satisfied at $L \approx 10000\ \text{km}, \quad \rho \approx 4.5\text{g/cm}^3, \quad E \approx 6.7\ \text{GeV}.$





Application 2: Magic Baseline

$$P(\nu_\mu \to \nu_e) = 4\left|\tilde{U}_{e2}\right|^2 \left|\tilde{U}_{\mu 2}\right|^2 \sin^2 \frac{\tilde{\Delta}_{21}}{2} + 4\left|\tilde{U}_{e3}\right|^2 \left|\tilde{U}_{\mu 3}\right|^2 \sin^2 \frac{\tilde{\Delta}_{31}}{2}$$

$$+ 2\Re\left(\tilde{U}^*_{e3}\tilde{U}_{\mu 3}\tilde{U}_{e2}\tilde{U}^*_{\mu 2}\right)\left(4\sin^2\frac{\tilde{\Delta}_{21}}{2}\sin^2\frac{\tilde{\Delta}_{31}}{2} + \sin\tilde{\Delta}_{21}\sin\tilde{\Delta}_{31}\right)$$

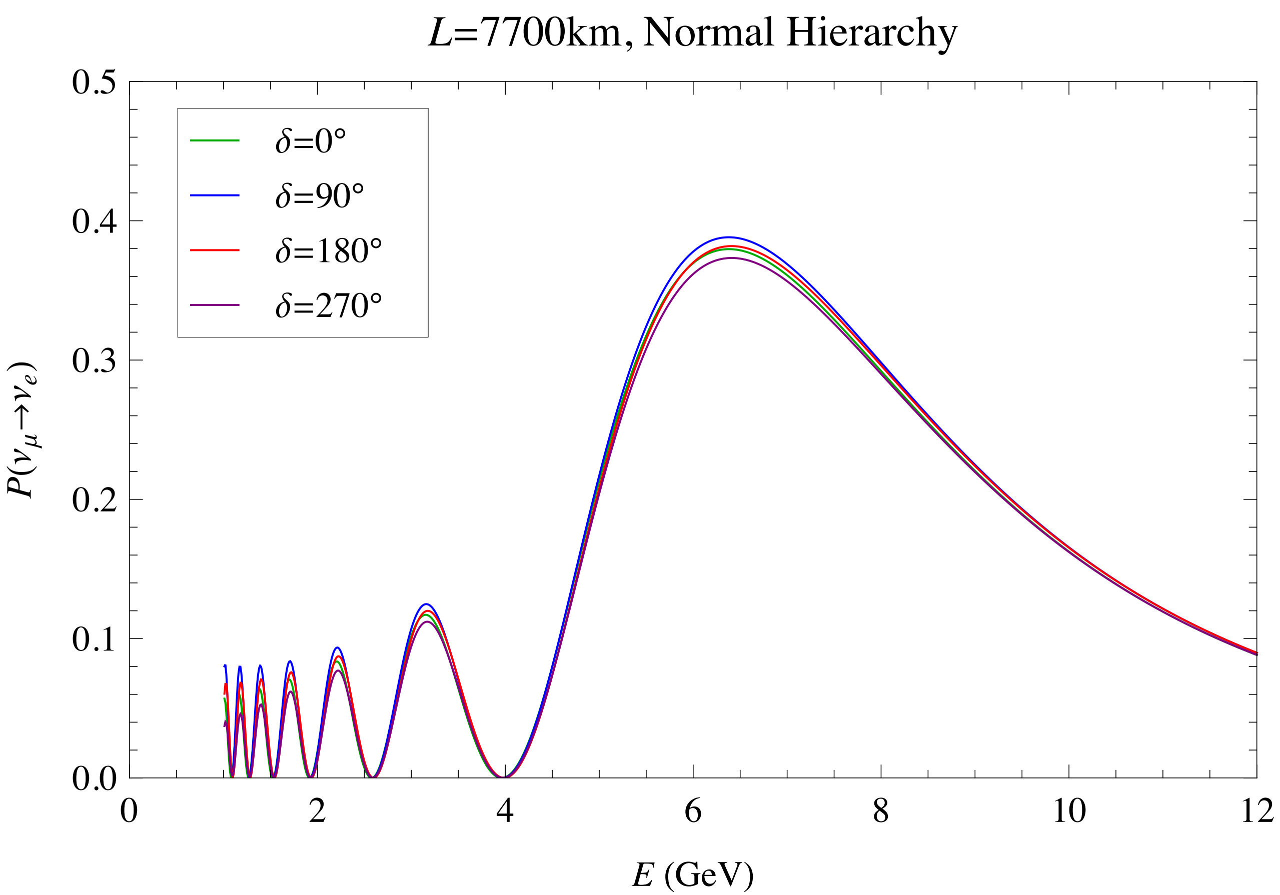
$$+ 4\tilde{J}_{(\mu,e)}\left(\sin^2\frac{\tilde{\Delta}_{21}}{2}\sin\tilde{\Delta}_{31} - \sin^2\frac{\tilde{\Delta}_{31}}{2}\sin\tilde{\Delta}_{21}\right)$$

To get rid of all dependence on the CP violating phase, we need to impose

$$\sin\frac{\tilde{\Delta}_{21}}{2} = 0 \quad \to \quad \frac{\delta\lambda_{21}}{4E}L \approx \frac{a}{4E}L \approx 10^{-4}\left(\frac{\rho}{\text{g/cm}^3}\right)\left(\frac{L}{\text{km}}\right) = \pi$$

This condition is satisfied around $L = 7700\ \text{km}$.

Application 2: Magic Baseline



New Physics

$$H_a$$

$$= U\begin{bmatrix} 1 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + \begin{bmatrix} a_e & 0 & 0 \\ 0 & a_\mu & 0 \\ 0 & 0 & a_\tau \end{bmatrix}$$

$$= U\begin{bmatrix} 1 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + \left(a_e - \frac{a_\mu + a_\tau}{2} \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \left(\frac{a_\mu - a_\tau}{2} \right) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \cdots$$

$$= U\begin{bmatrix} 1 & 0 & 0 \\ 0 & \delta m_{21}^2 & 0 \\ 0 & 0 & \delta m_{31}^2 \end{bmatrix} U^\dagger + a\begin{bmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & -\xi \end{bmatrix}, \qquad \xi << 1.$$

After the 2nd Rotation:

$$H_a''' = W^\dagger H_a'' W = \begin{bmatrix} \lambda_-' & -a(c_{12}'c_{13}s_{13} - \xi s_{12}')s_\phi & a(c_{12}'c_{13}s_{13} - \xi s_{12}')c_\phi \\ -a(c_{12}'c_{13}s_{13} - \xi s_{12}')s_\phi & \lambda_\mp'' & 0 \\ a(c_{12}'c_{13}s_{13} - \xi s_{12}')c_\phi & 0 & \lambda_\pm'' \end{bmatrix},$$

Third Rotation:

Rotation angle $\approx \dfrac{a\xi}{\delta m_{31}^2}$

Normal Hierarchy $\rightarrow$ 12 rotation $\rightarrow$ can be absorbed into θ_{12}

Inverted Hierarchy $\rightarrow$ 13 rotation $\rightarrow$ can be absorbed into θ_{23}

After the 2nd Rotation:

δ=0 case

$$\tilde{U} = UQVWY$$

$$= R_{23}(\theta_{23},0)R_{13}(\theta'_{13},0)R_{12}(\theta'_{12},0)R_{13}(\chi,0)$$

$$\approx R_{23}(\theta_{23},0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}R_{13}(\chi,0)$$

$$= R_{23}(\theta_{23},0)R_{23}(-\chi,0)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= R_{23}(\theta_{23} - \chi,0)R_{13}(\theta'_{13},0)R_{12}(\theta'_{12},0)$$

Where does the extra shift appear?

$$P(\nu_\mu \to \nu_\mu) \approx 1 - \sin^2[2(\theta_{23} + \chi)]\sin^2 \frac{\tilde{\Delta}_{21}}{2} \qquad \text{(Normal Hierarchy)}$$

$$\approx 1 - \sin^2[2(\theta_{23} - \chi)]\sin^2 \frac{\tilde{\Delta}_{31}}{2} \qquad \text{(Inverted Hierarchy)}$$

Conclusion:

- The Jacobi diagonalization procedure allows up to derive compact, transparent, and accurate analytical approximations to neutrino oscillation probabilities in matter.