The subjects: Reformulation and some generalizations

Mickelsson's twisted K-theory invariant and its generalizations

①

topological K-theory w/ local coefficients

[Donovan-Karoubi, Rosenberg]

(home of RR charges of D-branes w/ background B-fields)

invariant of some odd twisted K-classes on 3-mflds

detects some torsions

(Chern character detects $\begin{cases} \text{all non-torsions} \\ \text{no torsions} \end{cases}$)

Thm 1 (reformulation)

For any principal $PU(H)$-bundle P on a mfd M, $\exists$ natural ~~iso~~ hom.

$$\mu_3 : \operatorname{Ker} \mu_1 \to H^3(M, \mathbb{Z}) / (\mathrm{Tor} + h(P) \cup H^0(M, \mathbb{Z})).$$

This recovers Mickelsson's invariant if M is cpt, ori, conn. and 3-dim.. $\blacksquare$

notation.

- $K^1_P(M)$ the odd twisted K-grp of M.

$$\cong \Gamma(M, P \times_{PU(H)} U_1(H)) / \text{homotopy}$$

$$\left(\begin{array}{l} P \to M \quad \text{principal } PU(H)\text{-bundle} \\ PU(H) = U(H)/U(1) \quad \text{the projective unitary grp of a separable } \infty\text{-dim Hilbert sp } H. \\ \ \circlearrowleft \text{conj} \\ U_1(H) = \{ g \in U(H) : g - 1 \text{ trace class} \} \xrightarrow{\det} U(1) \end{array} \right)$$

- $\mu_1 : K^1_P(M) \to H^1(M, \mathbb{Z}) \cong C^\infty(M, U(1)) / \text{htpy}$

$$[g] \longmapsto [\det(g)]$$

- $h(P) \in H^3(M, \mathbb{Z}) \cong \{\text{principal } PU(H)\text{-bundle} / M\} / \text{iso}$

Ex $M = S^3$. $h = h(P) \in H^3(M, \mathbb{Z}) \cong \mathbb{Z}$, $h \neq 0$

$$\mu_3 : \operatorname{Ker}\mu_1 \longrightarrow \mathbb{Z}/h\mathbb{Z} \quad \text{(iso)}$$

$$\operatorname{Ker}\mu_1 = K^1_P(S^3)$$

original construction: section $g \rightsquigarrow \{ g_i : U_i \to U(H) \}$, $M = \bigcup U_i$

$\mu_3([g]) \in \mathbb{Z}/h$ is defined by integrating $\frac{-1}{24\pi^2} \operatorname{tr}[(g_i^{-1} dg_i)^3]$

and adding correction terms.

idea of reformulation

consider Čech–de Rham cocycle involving the 3-forms..

idea of generalization

use $\omega_{2k+1} = \operatorname{tr}[(g_i^{-1} dg_i)^{2k+1}]$.

Thm 2 (generalization) $\forall P \to M$.

$\exists \mu_5 : \operatorname{Ker}\mu_3 \xrightarrow{\text{hom}} H^5(M, \mathbb{R}) / h(P) \cup H^2(M, \mathbb{R})$ $(\leftrightarrow \omega_5)$

$\exists \nu_4 : K^1_P(M) \xrightarrow[\text{map}]{} H^4(M, \mathbb{R})$ $(\leftrightarrow \omega_3 \omega_1)$

$\exists \nu_9 : K^1_P(M) \longrightarrow H^9(M, \mathbb{R})$ $(\leftrightarrow \omega_5 \omega_3 \omega_1)$

μ_3, μ_5 are related to the Atiyah–Hirzebruch spectral sequence.

ν_4, ν_9 generate non-trivial characteristic classes for $K^1_P(M)$

(
§ Some details of construction

§ Relation to AHSS

§ Char. Class.
)

§ Some details of constructions

∘ To construct a Čech-de Rham 3-cocycle defining μ_3, choose:

① a good cover $M = \bigcup_i U_i$

② local sections $s_i \left(\begin{array}{c} P|_{U_i} \\ \downarrow \\ U_i \end{array} \right)$ $\quad$ ($\Rightarrow$ transition fct $\bar{\phi}_{ij} : U_{ij} \to PU(H)$)

③ lifts
$$\begin{array}{ccc} & & U(H) \\ & \overset{\phi_{ij}}{\nearrow} & \downarrow \\ U_{ij} & \xrightarrow[\bar{\phi}_{ij}]{} & PU(H) \end{array}$$
($\Rightarrow$ $f_{ijk} : U_{ijk} \to U(1)$ s.t. $g_{ij} g_{jk} = f_{ijk} g_{ik}$)

④ $\eta_{ijk} : U_{ijk} \to \mathbb{R}$ s.t. $f_{ijk} = \exp 2\pi\sqrt{-1}\, \eta_{ijk}$

($\Rightarrow$ $h_{ijkl} := (\delta\eta)_{ijkl} = \eta_{jkl} - \eta_{ikl} + \eta_{ijl} - \eta_{ijk} \in \mathbb{Z}$

$h(P) = [(h)] \in H^3(M, \mathbb{Z})$)

⑤ Given $g \in \Gamma(M, P \times_{PU(H)} U_1(H))$ s.t. $[g] \in \operatorname{Ker} \mu_1$,

$\alpha : M \to \mathbb{R}$ s.t. $\det(g) = \exp 2\pi\sqrt{-1}\, \alpha$.

∘ Čech-de Rham 3-cocycle

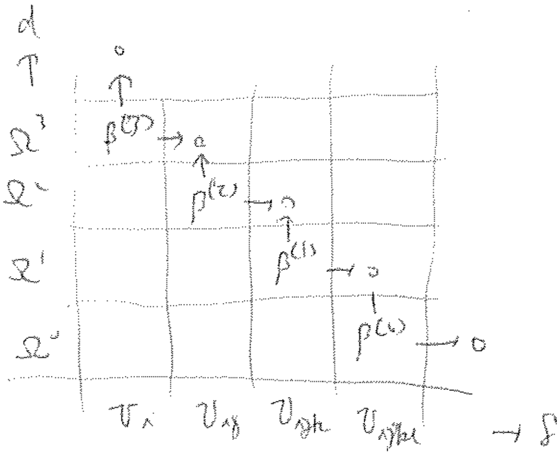
$g_i \leftarrow (s_i, g)$

$$\beta^{(3)}_i = \frac{-1}{24\pi^2} \operatorname{tr}[(g_i^{-1} dg_i)^3]$$

$$\beta^{(2)}_{ij} = -\frac{1}{8\pi^2} \{ \operatorname{tr}[g_j^{-1} dg_j \wedge d\phi_{ji} \phi_{ji}^{-1}] - \operatorname{tr}[\phi_{ji}^{-1} d\phi_{ji} \wedge dg_i g_i^{-1}] \}$$

$$\beta^{(1)}_{ijk} = -\eta_{ijk}\, d\alpha$$

$$\beta^{(0)}_{ijkl} = -h_{ijkl}\, \alpha$$



$\rightsquigarrow$ $\mu_3^{\mathbb{R}} : \operatorname{Ker} \mu_1 \to H^3(M, \mathbb{R}) / h(P) \cup H^0(M, \mathbb{Z})$ $\quad$ to get a well-defined map indep. of ⑤

- Generally, ~~the same~~ a similar construction gives

$$K^1_p(M) \to H^3(M,\mathbb{R})/H^3(M,\mathbb{Z}).$$

But this map is trivial. (by some ~~knowledge~~ knowledge about char. class)

$$\begin{tikzcd}
 & & \operatorname{Ker}\mu_3 \arrow[r, hook] \arrow[d, "\mu_3^{\mathbb{R}}"] \arrow[dll, dashed, "\mu_3"'] & K^1_p(M) \arrow[d] \\
0 \arrow[r] & \dfrac{H^3(M,\mathbb{Z})}{\mathrm{Tor} + h(p)\cup H^0_{\mathbb{Z}}} \arrow[r] & \dfrac{H^3(M,\mathbb{R})}{h(p)\cup H^0_{\mathbb{Z}}} \arrow[r] \arrow[ur, phantom, "\circlearrowleft"] & \dfrac{H^3(M,\mathbb{R})}{H^3(M,\mathbb{Z})}
\end{tikzcd}$$

exact

- the case of μ_T

1. For g, we just get a cochain.
2. If $[g] \in \operatorname{Ker}\mu_3$, ~~a choice of~~ then a choice of a Čech-deRham 2-cochain trivializing $\mu_3([g])$ serves ~~as~~ correction terms for the cochain to be a 5-cocycle...
3. To get a well-defined map indep of the choice above, we must divide $H^5(M,\mathbb{R})$ by $h(p)\cup H^2(M,\mathbb{R})$

- the case of ν_4 and ν_q : ~~no~~ ~~easier~~ easier

§ Rel to AHSS

AHSS [Atiyah-Segal]

$\{ d_r : E_r^{p,q} \to E_r^{p+1, q-r+1} \}$ a sequence of cochain complexes s.t.

① $E_{r+1}^{p,q} = \mathrm{Ker}(d_r : E_r^{p,q} \to E_r^{p+1,q-r+1}) / \mathrm{Im}\, d_r$

② $E_2^{p,q} = H^p(M, K^q(pt))$

③ $\exists r \gg 1$ s.t. $E_r = E_{r+1} = E_{r+2} = \cdots =: E_\infty$ if ~~dim M <~~ M cpt ∞

④ $0 \to F^{2k+3} K_p^1(M) \to F^{2k+1} K_p^1(M) \to E_\infty^{2k+1, 0} \to 0$ exact

where $F^{2k+1} K_p^1(M) = \mathrm{Ker}[\, \mathrm{res} : K_p^1(M) \to K_p^1(M_{<2k+1}) \,]$

regard as a CW cpx

the union of cells of dim $< 2k+1$

Prop

1. μ_1 agrees w/ the composition $K_p^1(M) \twoheadrightarrow E_\infty^{1,0} \hookrightarrow H^1(M, \mathbb{Z})$.

$(\Rightarrow \mathrm{Ker}\, \mu_1 = F^3 K_p^1(M))$

2. μ_3 agrees w/ the composition of

$$F^3 K_p^1(M) \twoheadrightarrow E_\infty^{3,0} \hookrightarrow \frac{H^3(M, \mathbb{Z})}{h(p) \cup H^0_{\mathbb{Z}}} \to \frac{H^3(M, \mathbb{Z})}{\mathrm{Tor} + h(p) \cup H^0_{\mathbb{Z}}}$$

$(\Rightarrow \mathrm{Ker}\, \mu_3 \supset F^5 K_p^1)$

3. $\mu_5|_{F^5 K_p^1}$ agrees w/ the composition of

$$F^5 K_p^1(M) \twoheadrightarrow E_\infty^{5,0} \hookrightarrow \frac{H^5(M, \mathbb{Z})}{h(p) \cup H^2_{\mathbb{Z}}} \to \frac{H^5(M, \mathbb{R})}{h(p) \cup H^2_{\mathbb{R}}}.$$

Key to the prop

- construction of AHSS

- $K^1(S^{2k+1}) \cong \mathbb{Z}$ can be realized by

$$[g] \longmapsto \int_{S^{2k+1}} \omega_k \times (\mathrm{const}).$$

application:

② $h(p) \in H^3(SU(3), \mathbb{Z}) \cong \mathbb{Z}$ odd

$\Rightarrow K_p^1(SU(3)) \cong \mathbb{Z}/h$ (known result)

① M: cpt ori 3-dim

$\Rightarrow \mu_3$ iso

§ Char. class.

<u>Def</u> A char class for odd twisted K-theory is a natural map

$K^1_P(M) \to H^*(M, \mathbb{R})$ defined for any $P \to M$

※. The K^1-version of an argument by Atiyah-Segal gives :

<u>Prop</u> $J^n :=$ the vector sp. gen. by char class $K^1_P(M) \to H^n(M, \mathbb{R})$. Then,

$$\sum_{n \geq 0} (\dim J^n) t^n = t^2 t^3 + (1 - t^2) \prod_{i \geq 0} (1 + t^{2i+1})$$

$$= 1 + t + t^3 + t^4 + t^8 + t^9 + t^{12} t \cdots$$

$$\begin{array}{cccc} \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ \mu_1 & h(P) & \nu_4 & \nu_9 \\ \updownarrow & & \updownarrow & \updownarrow \\ \omega_1 & & \omega_3\omega_1 & \omega_5\omega_3\omega_1 \end{array}$$

The argument implies :

- Combinations like $\omega^5\omega^1$ do not give non-trivial char. classes.

- $t^8 \leftrightarrow \frac{1}{7}\omega^7\omega^1 - \frac{1}{3}\omega^5\omega^3$