Topological defects and generalised orbifolds

Nils Carqueville

Simons Center

Introduction

Motivation

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$$\Rightarrow K_t(z,y) = \int \mathrm{d}x_1\, K_{t-t_1}(z,x_1)K_{t_1}(x_1,y)$$

$$= \int \mathrm{d}x_1 \dots \mathrm{d}x_n\, K_{t-t_n}(z,x_n)K_{t_n-t_{n-1}}(x_n,x_{n-1})\dots K_{t_1}(x_1,y)$$

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Similarly in **quantum field theory**

$$K_\Sigma(\phi_{\text{out}}, \phi_{\text{in}}) = \int_{\phi|_{\partial_{\text{in}}\Sigma}=\phi_{\text{in}}}^{\phi|_{\partial_{\text{out}}\Sigma}=\phi_{\text{out}}} \mathcal{D}\phi \, e^{-S[\phi]}$$

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Motivation

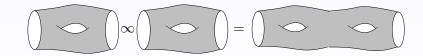"Evolution" Σ gives rise to map between spaces of quantum states:

$$Z_\Sigma : \mathcal{H}_{\mathsf{in}} \longrightarrow \mathcal{H}_{\mathsf{out}} \,, \qquad \big(Z_\Sigma(\eta)\big)(\phi_{\mathsf{out}}) = \int \mathcal{D}\phi_{\mathsf{in}} \, K_\Sigma(\phi_{\mathsf{out}}, \phi_{\mathsf{in}})\eta(\phi_{\mathsf{in}})$$

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In general, additivity property $U_{t_1} \circ U_{t_2} = U_{t_1+t_2}$ turns into

gluing axiom $\qquad Z_{\Sigma_1 \infty \Sigma_2} = Z_{\Sigma_1} \circ Z_{\Sigma_2}$

Categories (are really easy)

A (small) **category** $\mathcal{C}$ has

- a set of *objects* $\mathrm{Ob}(\mathcal{C})$,
- for every $A, B \in \mathrm{Ob}(\mathcal{C})$ a set of *morphisms* $\mathrm{Hom}(A, B)$, with special elements $1_A \in \mathrm{Hom}(A, A)$,
- for every $A, B, C \in \mathrm{Ob}(\mathcal{C})$ a map

$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \longrightarrow \mathrm{Hom}(A, C)$$

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Examples.

- points and arrows in directed graphs
- vector spaces and linear maps
- manifolds and diffeomorphisms
- branes and open strings in topological string theory

Two-dimensional topological field theory

A closed **topological field theory** is a map

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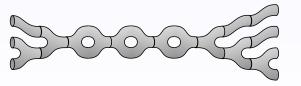A closed **topological field theory** is a map

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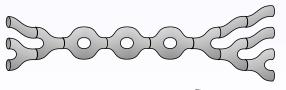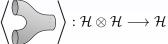   morphisms: smooth surfaces between circles

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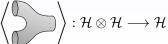$$\langle - \rangle : \mathrm{Bord} \longrightarrow \mathrm{Vect}$$

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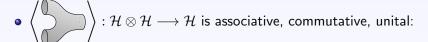
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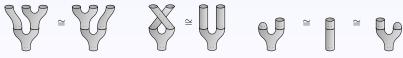
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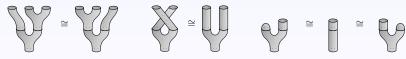
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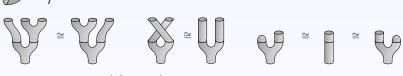

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- **B-twisted Landau-Ginzburg models**: Let $W \in \mathbb{C}[x_1, \ldots, x_n]$.

$$\mathcal{H} = \mathbb{C}[x_1, \ldots, x_n]/(\partial W) \,, \qquad (f, g) \longmapsto \mathrm{Res} \left[\frac{fg \, \mathrm{d}x}{\partial_{x_1} W \ldots \partial_{x_n} W} \right]$$

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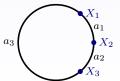
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Operator product of fields, unit = identity field

$$\left\langle \vcenter{\hbox{$\bullet \, \psi \atop \bullet \, \varphi$}} \right\rangle = \left\langle \vcenter{\hbox{$\bullet \, \psi\varphi$}} \right\rangle = \left\langle \vcenter{\hbox{$\bullet \, 1 \atop \bullet \, \psi\varphi$}} \right\rangle$$

Bicategories (pretty easy too)

Definition. A **bicategory** $\mathcal{B}$ has

- objects $a, b, \ldots$

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- functors $\otimes : \mathcal{B}(b, c) \times \mathcal{B}(a, b) \longrightarrow \mathcal{B}(a, c)$

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Examples of bicategories:

- **WZW models**: bundle gerbes
- **A-models**: symplectic manifolds, Lagrangian correspondences
- **B-models**: varieties, Fourier-Mukai kernels
- **Landau-Ginzburg models**: potentials, matrix factorisations
- categories, functors, natural transformations
- algebras, bimodules, bimodule maps

Davydov/Kong/Runkel, Fuchs/Runkel/Schweigert/Suszek/Waldorf, Căldăraru/Willerton, Wehrheim/Woodward, Carqueville/Murfet

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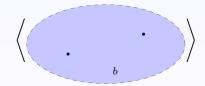**Everything about theory b can be recovered from defect A.**
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Carqueville/Runkel 2012

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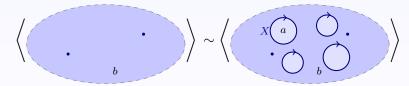Think of A as '**generalised symmetry**', replacing orbifold group.

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generalised orbifolds

equivalant completion

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Hochschild (co)homology

2d TFT^{or} with defects

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equivariant completion

Outline

Duality and adjunctions

Bicategories

Definition. A **bicategory** $\mathcal{B}$ has
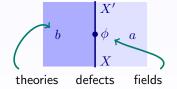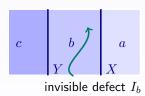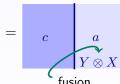
- theories
  objects $a, b, \dots$
- categories $\mathcal{B}(a, b)$ of 1-morphism $X, Y, \dots$ (defects) and 2-morphisms $\phi, \psi, \dots$ (fields)
- fusion
  functors $\otimes : \mathcal{B}(b, c) \times \mathcal{B}(a, b) \longrightarrow \mathcal{B}(a, c)$
- invisible defects
  units $I_a \in \mathcal{B}(a, a)$ and for $X \in \mathcal{B}(a, b)$ natural isomorphisms
  $$I_b \otimes X \cong X, \quad X \otimes I_a \cong X, \quad (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$$

theories   defects   fields   invisible defect I_b fusion

Diagrammatics in bicategories

$$X$$

$$\Big| = 1_X$$

$$X$$

Diagrammatics in bicategories

$$X \quad\quad\quad Y$$

$$\Big| \quad = 1_X \quad\quad \varphi \bullet \quad = \varphi : X \longrightarrow Y$$

$$X \quad\quad\quad X$$

Diagrammatics in bicategories

$$\Big| = 1_X \qquad \varphi\,\big|_\bullet = \varphi : X \longrightarrow Y \qquad \begin{matrix}\psi\,\bullet \\ \varphi\,\bullet\end{matrix} = \psi\varphi$$

$$X \qquad\qquad Y \qquad\qquad\qquad Z$$
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Diagrammatics in bicategories

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$$\Big|_{X}^{X} = 1_X \qquad \varphi \bullet \Big|_{X}^{Y} = \varphi : X \longrightarrow Y \qquad \psi \bullet \atop \varphi \bullet \Big|_{X}^{Z} = \psi\varphi \qquad \varphi \bullet \quad \bullet \varphi' \atop {}_{X\ X'}^{Y\ Y'} = \varphi \otimes \varphi'$$

$$\phi \bullet \atop {}_{X\ Y}^{Z} = \phi : X \otimes Y \longrightarrow Z$$

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Always read diagrams from bottom to top and from right to left.

Diagrammatics in bicategories

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$$\overset{Z}{\underset{X \; Y}{\phi}} = \phi : X \otimes Y \longrightarrow Z \qquad \rho_X \overset{X}{\underset{X \quad I}{\bullet}} \qquad \overset{X}{\underset{I \quad X}{\bullet}} \lambda_X$$

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Orientation matters:
$$b \quad \overset{X}{\underset{X}{\uparrow}} \quad a$$

Diagrammatics in bicategories

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Orientation and adjoints

$$\bigcap \;=\; \mathrm{ev}_X : X^\dagger \otimes X \longrightarrow I \qquad\qquad \bigcup \;=\; \mathrm{coev}_X : I \longrightarrow X \otimes X^\dagger$$

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Defects are **topological**:

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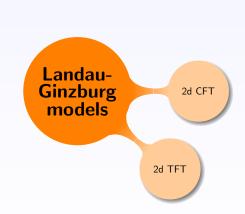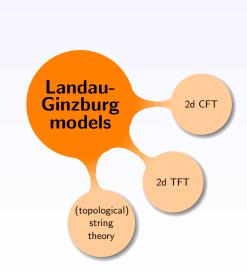
Definition. A bicategory **has adjoints** if for each 1-morphism X there is a 1-morphism $X^\dagger$ with 2-morphisms $\mathrm{ev}_X, \mathrm{coev}_X$ such that the above *Zorro moves* hold.

THE LEGEND Z OF ZORRO

Landau–Ginzburg models

Landau-Ginzburg models

2d CFT

2d TFT

(topological) string theory

Landau-Ginzburg models

2d CFT

2d TFT

matrix factorisations

(topological) string theory

Landau-Ginzburg models

- homological knot invariants
- homological mirror symmetry
- Calabi-Yau geometry
- singularity theory
- matrix factorisations
- (topological) string theory
- 2d TFT
- 2d CFT

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Kontsevich, Kapustin/Li 2002, Lazaroiu 2003, Brunner/Roggenkamp 2007

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- fields between X and Y: BRST cohomology of

$$\operatorname{Hom}(X, Y) \ni \psi \longmapsto d_Y \psi - (-1)^{|\psi|} \psi d_X$$

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- operator product: matrix multiplication

Brunner/Roggenkamp 2007, Kapustin/Rozansky 2004

First main result

Theorem.   The bicategory $\mathcal{LG}$ of Landau-Ginzburg models has adjoints.

Carqueville/Murfet 2012, see also Brunner/Roggenkamp/Rossi 2009, Carqueville/Runkel 2010

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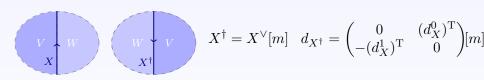(Note: we suppress various signs)

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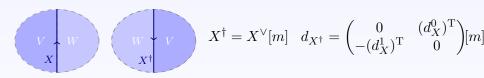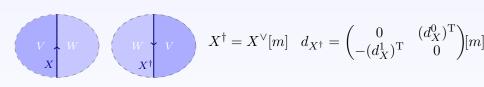Let $W \in k[x_1, \ldots, x_n]$, $V \in k[z_1, \ldots, z_m]$, X matrix fact. of $V - W$:

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Proof: homological perturbation, associative Atiyah classes

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$\mathrm{ch}(X) := \beta^X(1)$ is the **boundary state** or **Chern character**

Applications

Theorem. The **Cardy condition** holds in $\mathcal{LG}$

Polishchuk/Vaintrob 2010, Carqueville/Murfet 2012

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Theorem. The **Cardy condition** holds in $\mathcal{LG}$: for matrix factorisations X, Y of W and maps $\Phi : X \longrightarrow X$, $\Psi : Y \longrightarrow Y$ we have

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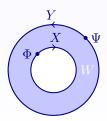
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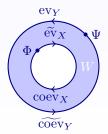
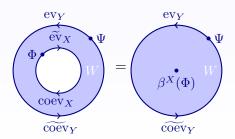where ${}_\Psi m_\Phi$ sends $\alpha : X \longrightarrow Y$ to $\Psi\alpha\Phi$.

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Generalised orbifolds

Generalised orbifolds

Main result. Let $X \in \mathcal{B}(a,b)$ have invertible $\dim(X)$, set $A = X^\dagger \otimes X$.
Everything about theory b can be recovered from defect A.
Think of A as '**generalised symmetry**', replacing orbifold group.

Idea. Introducing X-bubbles in b-correlator is scaling by $\dim(X)$.
Blowing up all X-bubbles produces a-correlator with A-defect network.

Generalised orbifolds

Let $\mathcal{B}$ be pivotal bicategory with adjoints. **Orbifold completion** $\mathcal{B}_{\mathrm{orb}}$

Carqueville/Runkel 2012, Fröhlich/Fuchs/Runkel/Schweigert 2009, (Fjelstad/Fröhlich/)Fuchs/Runkel/Schweigert 200x

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If $\dim(X)$ is *invertible* (easy to check for $\mathcal{B} = \mathcal{LG}$) then:

- $A = X^\dagger \otimes X$ is a also *separable*.
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Holds for *any* pivotal bicategory $\mathcal{B}$, e. g. also for B-models – expect generalisation of homological mirror symmetry!

Generalised orbifolds: Landau-Ginzburg examples

- "ordinary" orbifolds: for finite symmetry group G of W we have

$$\mathrm{hmf}(W)^G = \mathcal{LG}(0, W)^G \cong \mathrm{mod}\Big(\bigoplus_{g \in G} I_g\Big)$$

Carqueville/Runkel 2012, see also Ashok/Dell'Aquila/Diaconescu 2004, Brunner/Roggenkamp 2007

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Task. Classify all defects with invertible quantum dimensions!

Carqueville/Runkel 2012, Brunner/Carqueville/Plencner 2013

Equivariant completion
and supersymmetry

Generalised orbifolds

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What if the algebra is not symmetric?

Lemma. A is symmetric iff its **Nakayama automorphism**

is the identity.

Generalised orbifolds

Fix bicategory with adjoints $\mathcal{B}$. **Orbifold completion** $\mathcal{B}_{\mathrm{orb}}$:
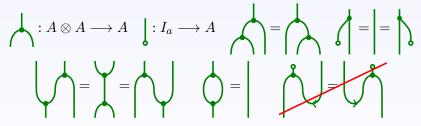
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"NS" and "R" sectors

Motivation. IR fixed points of Landau-Ginzburg models have equivalence

{ RR ground states } $\cong$ { (c,c) fields }

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Let $(a, A) \in \mathcal{B}_{\mathrm{eq}}$. Consider $\mathbb{C}$-algebra $\mathrm{Hom}(I_a, A)$ with

$$\alpha \cdot \beta = \bigwedge_{\alpha \quad \beta} \qquad \left\langle \alpha, \beta \right\rangle_{(a,A)} = \left\langle \bigwedge_{\alpha \quad \beta} \right\rangle_a$$

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Two ways to "wrap A around α" give *projectors*

$$\pi_A^{(\mathrm{c,c})} : \alpha \longmapsto \; \text{<image>} \qquad \pi_A^{\mathrm{RR}} : \alpha \longmapsto \; \text{<image>}$$

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For ordinary LG orbifolds ($A = A_G$) their images recover (c,c) fields and RR ground states!

Brunner/Carqueville/Plencner 2013

"NS" and "R" sectors

Theorem. $\pi_A^{(c,c)}$ and π_A^{RR} compute **Hochschild (co)homology**:

$$\text{im}(\pi_A^{(c,c)}) \cong \text{HH}^\bullet(A) = \text{Hom}_{AA}(A, A)$$
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So $\mathrm{HH}^\bullet(A)$ and $\mathrm{HH}_\bullet(A)$ are the special cases $n = 0$ and $n = 1$ of

$$\mathcal{H}_A^n = \mathrm{im}\left(\begin{array}{c} \alpha \end{array} \longmapsto \gamma_A^n \begin{array}{c} \alpha \end{array}\right) \qquad n \in \mathbb{Z}$$

Brunner/Carqueville/Plencner 2013

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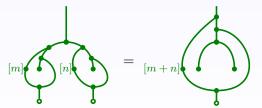
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Evidence: $\mathcal{H}_A^m \times \mathcal{H}_A^n \longrightarrow \mathcal{H}_A^{m+n}$ as

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Outlook.

- more generalised orbifolds
- generalisation of homological mirror symmetry
- refined link invariants: compute $\mathfrak{so}(2n)$ from $\mathfrak{sl}(n)$

Summary

extended
2d TFT[fr]

Hochschild
(co)homology

2d TFT[or]
with defects

supersymmetry

generalised
orbifolds

equivariant
completion