Introduction to Saito's theory of primitive forms

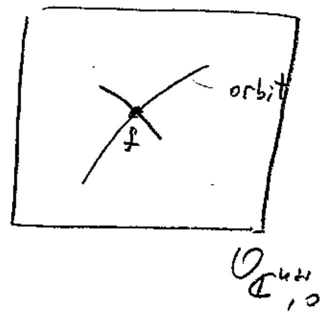
PART I: definition

$f \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ s.t.

$$H = \mathcal{O}_{\mathbb{C}^{n+1},0} / (f_{x_0}, \dots, f_{x_n}) \ , \quad x = (x_0, \dots, x_n) \text{ linear coords. on } \mathbb{C}^{n+1}$$

is a finite dimensional vector space, i.e., 0 is an isolated critical point.

Let $\{\phi_i\}_{i=1}^{\mu}$ be germs in $\mathcal{O}_{\mathbb{C}^{n+1},0}$ that project to a basis of H; then



$$F(x,s) := f(x) + \sum_{j=1}^{\mu} s_j \phi_j(x)$$

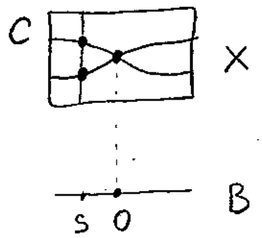
1. The Kodaira–Spencer map

$$X := \{ (x, s, \lambda) \in B_\rho^{n+1} \times B_\eta^{\mu} \times B_\delta^1 \mid \lambda = F(s,x) \}$$

(η, ρ, δ) are sufficiently small and chosen appropriately.

$$C = \{ F_{x_0} = \dots = F_{x_n} = 0 \} \subseteq X \quad \text{critical set of } F$$

$$\begin{array}{ccc} C & \hookrightarrow & X \\ & {\scriptstyle q_c} \searrow & \downarrow q \\ & & B := B_\eta^{\mu} \end{array}$$



$q_c^{-1}(0) = \{0\} \Rightarrow$ by decreasing η, ρ, and δ we may assume that q_c is a finite map. Put $\mathcal{O}_C = \mathcal{O}_X / (F_{x_0} \dots F_{x_n})$

Theorem 1. If (η, ρ, δ) are appropriately chosen then the map (Kodaira–Spencer map)

$$T_B \longrightarrow q_* \mathcal{O}_C \ , \quad \frac{\partial}{\partial s_i} \longmapsto \frac{\partial F}{\partial s_i} \bmod (F_{x_0} \dots F_{x_n})$$

is an isomorphism.

Pf. $\mathfrak{m} \subset \mathcal{O}_{B,0}$ ideal of $0 \Rightarrow T_{B,0}/\mathfrak{m}\, T_{B,0} \xrightarrow{\cong} (q_* \mathcal{O}_C)_0 / \mathfrak{m}\, (q_*\mathcal{O}_C)_0 \cong H$

Nakayama's lemma implies that $T_{B,0} \cong (q_* \mathcal{O}_C)_0$

(or flatness of $T_{B,0}$ and $(q_*\mathcal{O}_C)_0$ as $\mathcal{O}_{B,0}$-modules)

Enough to prove that $T_{B,0} \to (q_* \mathcal{O}_C)_0$ is an isomorphism.

$R := \mathcal{O}_{B,0}$, $\mathfrak{m} = (s_1, \dots, s_\mu)\mathcal{O}_{B,0}$ is a local Noetherian ring and $M := T_{B,0}$, $N := (q_* \mathcal{O}_C)_0$ are flat fin. generated R-modules. Since flat, they must be free of rank respectively $\dim_{\mathbb{C}}(M/\mathfrak{m}M)$ and $\dim_{\mathbb{C}}(N/\mathfrak{m}N)$.

But $M/\mathfrak{m}M \cong \mathbb{C}^\mu \longrightarrow N/\mathfrak{m}N \cong H$

is an isomorphism by definition $\Rightarrow \dim_{\mathbb{C}}(M/\mathfrak{m}M) = \dim_{\mathbb{C}}(N/\mathfrak{m}N)$.

Look at the map $\varphi: M \to N$ s.t. $\overline{\varphi}: M \otimes R/\mathfrak{m} \to N \otimes R/\mathfrak{m}$

is an isomorphism. Put $M' = \varphi(M)$

$$\begin{array}{ccc} M \otimes R/\mathfrak{m} & \xrightarrow{\overline{\varphi}} & N \otimes R/\mathfrak{m} \\ \| & & \| \\ M/\mathfrak{m}M & \xrightarrow{\cong} & N/\mathfrak{m}N \end{array}$$

$M/\mathfrak{m}M \searrow M' \otimes_R R/\mathfrak{m} \nearrow N/\mathfrak{m}N$ injective

$0 \leftarrow M' \leftarrow M \leftarrow K \leftarrow 0$

$0 \leftarrow M' \otimes_R R/\mathfrak{m} \leftarrow M \otimes_R R/\mathfrak{m} \leftarrow K \otimes_R R/\mathfrak{m}$

$N' = N/M'$, i.e.,

$0 \leftarrow N' \leftarrow N \leftarrow M' \leftarrow 0$ $\quad \otimes_R R/\mathfrak{m}$

$0 \leftarrow N' \otimes_R R/\mathfrak{m} \leftarrow N \otimes_R R/\mathfrak{m} \xleftarrow{\text{injective}} M' \otimes_R R/\mathfrak{m} \leftarrow$

$\leftarrow \mathrm{Tor}_1^R(N', R/\mathfrak{m}) \leftarrow 0 \leftarrow \mathrm{Tor}_1(M', R/\mathfrak{m}) \leftarrow \mathrm{Tor}_2^R(N', R/\mathfrak{m})$

$\mathrm{Tor}_2^R(N', R/\mathfrak{m}) = 0$ because N' is flat

$\mathrm{Tor}_1(M', R/\mathfrak{m}) = 0$ and M' is flat

$\mathrm{Tor}_1^R(N', R/\mathfrak{m}) = 0$ i.e. N' is flat

since $\dim_{\mathbb{C}}(M/\mathfrak{m}M) = \dim_{\mathbb{C}}(N/\mathfrak{m}N)$ we must have $M' \otimes_R R/\mathfrak{m} = N \otimes_R R/\mathfrak{m}$

$\Rightarrow N' \otimes_R R/\mathfrak{m} = 0$ i.e. $N' = 0$ $\Rightarrow M' = N$ i.e. φ is surjective.

Similarly $M \otimes_R R/\mathfrak{m} \xrightarrow{\cong} M' \otimes_R R/\mathfrak{m}$ so $K \otimes_R R/\mathfrak{m} = 0$ i.e. $K = 0 \Rightarrow$

φ is injective. ☑

2. The period map

$$\mathcal{H}_F = \mathbb{H}^{n+1}\left(q_* \Omega^\bullet_{X/B}((z)),\ z d + dF\wedge \right) = q_*\, \mathbb{H}^{n+1}\left(\Omega^\bullet_{X/B}((z)),\ z d + dF\wedge \right)$$

$$\begin{tikzcd} X \subset B \times B^{n+1}_{\xi} \times B^1_{\zeta} \arrow[d, "q"'] \arrow[d, phantom] \\ B \end{tikzcd} \qquad X \xrightarrow{pr_1} B$$

if $U \subset B$ is an open polydisc then

$$\mathcal{H}_F(U) = \Omega^{n+1}_{X/B}(q^{-1}U)((z)) \Big/ \underbrace{(z d + dF\wedge)}_{= e^{-F/z} \circ z d \circ e^{F/z}} \Omega^n_{X/B}(q^{-1}U)((z))$$

$$\omega = g(x, s; z)\, dx_0 \wedge \dots \wedge dx_n$$

$$g(x,s) = \sum_{k=k_0}^{\infty} g_k(x,s) z^k, \qquad g_k \in \mathcal{O}_X(q^{-1}U)$$

We denote by

$$\int e^{F(s,x)/z}\, \omega \qquad \text{the equivalence class of } \omega \text{ in } \mathcal{H}_F(U).$$

<u>Def 1.</u> We say $\omega \in \Omega^{n+1}_{X/B}(X)$ is a <u>volume form</u> if

$$\omega = g(s,x)\, d^{n+1}x, \quad g \in \mathcal{O}_X(X), \quad g(s,x) \neq 0 \ \forall (s,x) \in X$$

The sheaf $\mathcal{H}_F$ is equipped w/ a connection (Gauss-Manin connection)

$$\nabla_\xi \int e^{F/z} g(x,s;z)\, d^{n+1}x = \int \xi \left(e^{F/z} g(x,s;z)\, d^{n+1}x \right)$$

$$\xi = \frac{\partial}{\partial s_i} \in T_B \quad \text{or} \quad \xi = \frac{\partial}{\partial z}.$$

<u>Lemma 1.</u> If ω is a volume form; then the $\overbrace{\text{map}}^{\mathbb{C}[[z]]\text{-modules}}$

$$\psi^{(0)} : T_B[[z]] \longrightarrow \underbrace{\mathcal{H}_F^{(0)}}_{\text{sections of } \mathcal{H}_F \text{ regular at } z=0 \text{ (no pole at } z=0)}$$

defined by $\xi \longmapsto z \nabla_\xi \int e^{F/z} \omega$, $\xi \in T_B$

is an isomorphism.

<u>Example:</u> A_2-singularity $\quad f(x) = \dfrac{x^3}{3}$

$$H = \mathcal{O}_{\mathbb{C},0} / x^2 \mathcal{O}_{\mathbb{C},0} \cong \mathbb{C}[x]/x^2$$

$$F(x,s) = \frac{x^3}{3} + s_1 x + s_2$$

$$\omega = dx$$

$$\int e^{F/z} x^3 dx = r^{(0)}(V), \quad V = ?$$

$$x \partial_x F = x^3 + s_1 \cdot x \quad \text{i.e.} \quad x^3 = -s_1 \cdot x + x F_x$$

$$\int e^{F/z}(-s_1 x + x F_x)\, dx = \int e^{F/z}(-s_1 x)\, dx +$$

$$+ \int e^{F/z} \cdot x \underset{\underset{-z\,dx}{\shortparallel}}{dF} = \left(-s_1\, z \nabla_{\frac{\partial}{\partial s_1}} - z^2 \frac{\partial}{\partial s_2}\right) \int e^{F/z}\omega$$

$$\Rightarrow \quad V = -s_1 \frac{\partial}{\partial s_1} - z \frac{\partial}{\partial s_2}.$$

3. The higher-residue pairing.

$U_i = \{F_{x_i} \neq 0\} \subset X$ Stein cover of X

$i = 0, 1, \dots, n$

$$\mathcal{H}_F^{(0)} \times \mathcal{H}_F^{(0)} \longrightarrow \mathcal{O}_B[\![z]\!]$$

Let $V \subset B$ be an open set, $q^{-1}(V) \cap U_i$, $i = 0, 1, \dots, n$

is a covering $\mathcal{U}$ of $q^{-1}(V) \setminus C$

<u>Lemma 2</u>. The sequence

$$0 \to C^p(\mathcal{U}, \hat{\Omega}^0_{X/B}) \xrightarrow{\hat{d}} C^p(\mathcal{U}, \hat{\Omega}^1_{X/B}) \xrightarrow{\hat{d}} \cdots \xrightarrow{\hat{d}} C^p(\mathcal{U}, \Omega^{n+1}_{X/B}) \to 0$$

is exact, where

$$\hat{\Omega}^i_{X/B} := \Omega^i_{X/B}[\![z]\!], \quad \hat{d} := z d + dF \wedge$$

and $p \geq 0$.

Let $*$ be the involution on $\mathcal{H}_F^{(0)}$ $z \longmapsto -z$

$$K_F(\omega_1,\omega_2) := \operatorname{Res}_{C/B}\left(\left(\left(\hat{d}^{-1}\check{\partial}\right)^n \hat{d}^{-1}\omega_1\right)\omega_2^*\right) \quad z^{n+1}$$

$$\frac{1}{(2\pi\sqrt{-1})^{n+1}} \int_{|F_{x_0}|=\cdots=|F_{x_n}|=\varepsilon}$$

Here $\omega_1, \omega_2 \in \Omega^{n+1}_{X/B}[\![z]\!]$ are arbitrary forms.

$\omega_1 \in C^0(\mathcal{U}, \hat{\Omega}^{n+1}_{X/B}) \Rightarrow (\hat{d}^{-1}\check{\partial})^n \hat{d}^{-1}\omega_1 \in C^n(\mathcal{U}, \Omega^0_{X/B}) =$

$= \mathcal{O}_X(U_0 \cap \cdots \cap U_n)$ i.e. function holom. on $X \setminus (\{F_{x_0}=0\} \cup \cdots \cup \{F_{x_n}=0\})$

Remark: Since $\hat{d} = e^{-F/z} z d\, e^{F/z}$ we have

$$\operatorname{Res}_{C/B} \cancel{e^{-F/z}}\left(\left((\cancel{z}d)^{-1}\check{\partial}\right)^n (\cancel{z}d)^{-1} e^{F/z}\omega_1\right)\left(e^{F/z}\omega_2\right)^* \quad \cancel{z^{n+1}}$$

Lemma 3. 1) $K_F(\omega_1,\omega_2) = (-1)^{n+1} K_F(\omega_2,\omega_1)^*$.

2) $p(z)\, K_F(\omega_1,\omega_2) = K_F(p(z)\omega_1,\omega_2) = K_F(\omega_1, p(-z)\omega_2)$.

3) $\xi\, K_F(\omega_1,\omega_2) = K_F(\nabla_\xi\omega_1,\omega_2) + K_F(\omega_1, \nabla_\xi\omega_2)$.

4. The primitive form

Assume that $\omega \in \Omega^{n+1}_{X/B}$ is a volume form

Put
$$K_F(\omega_1,\omega_2) = \sum_{p=0}^{\infty} K_F^{(p)}(\omega_1,\omega_2)\, z^{n+1+p}$$

Def 2. The form ω is called primitive if:

(i) $K_F^{(p)}\left(\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j}\right) = 0 \quad \forall p \geq 1 \quad \forall i,j$

(ii) $K_F^{(p)}\left(z^2 \nabla_{\frac{\partial}{\partial s_i}} \nabla_{\frac{\partial}{\partial s_j}} \omega,\ z\nabla_{\frac{\partial}{\partial s_k}}\omega\right) = 0 \quad \forall p \geq 2 \quad \forall i,j,k$

(iii) $K_F^{(p)}\left(z^2 \nabla_{\frac{\partial}{\partial z}} z\nabla_{\frac{\partial}{\partial s_i}} \omega,\ z\nabla_{\frac{\partial}{\partial s_j}}\omega\right) = 0 \quad \forall\, p \geq 2$ and $\forall\, i,j$. //

Assume that ω is a primitive form; then T_B has the following structures:

$$T_B \cong q_* \mathcal{O}_C \quad \Rightarrow \quad \text{multiplication } \bullet$$

$$\left(\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j}\right) := K_F^{(0)}\left(z\nabla_{\frac{\partial}{\partial s_i}}\omega,\ z\nabla_{\frac{\partial}{\partial s_j}}\omega\right) \quad \Rightarrow \text{symmetric bi-linear pairing}$$

$$= \mathrm{Res}_{C/B} \frac{\phi_i(x)\,\phi_j(x)}{F_{x_0} \cdots F_{x_n}}\, g^2(s,x)\, d^{n+1}x$$

The period isomorphism $\quad \pi : B \times \mathbb{C}^\times \to B$, $(s,z) \mapsto s$

$$r^{(0)} : T_B[[z]] \cong \mathcal{H}_F^{(0)}$$

induces a connection ∇ on $\pi^* T_B$:

$$\left|\begin{array}{l} r^{(0)}\left(z\nabla_W V\right) := z\nabla_W\, r^{(0)}(V) \qquad \forall\, v, w \in T_B \\ r^{(0)}\left(z^2 \nabla_{\frac{\partial}{\partial z}} V\right) = z^2 \nabla_{\frac{\partial}{\partial z}}\, r^{(0)}(V) \end{array}\right.$$

Conditions (i) and (ii) imply that ∇ takes the form:

$$\left|\begin{array}{l} \nabla_{\frac{\partial}{\partial s_i}} = \nabla^{L.C.}_{\frac{\partial}{\partial s_i}} + \frac{1}{z}\left(\frac{\partial}{\partial s_i}\bullet\right) \qquad \forall\, i = 1, 2, \ldots, \mu \\ \nabla_{\frac{\partial}{\partial z}} = \frac{\partial}{\partial z} + \frac{1}{z^2} E\bullet + \frac{1}{z}\left(\theta + c\cdot \mathrm{Id}\right) \end{array}\right.$$

where $E \in T_S \cong q_* \mathcal{O}_C$ corresponds to F (Euler vector field)

$$\theta(V) = \nabla^{L.C.}_V E$$

c is some constant which we can normalize as we wish.

$\Rightarrow (B, \bullet, (\cdot,\cdot), E, 1)$ gives a Frobenius structure. [Saito-Takahashi]

5. The vanishing cohomology

$$\begin{array}{ccc} X & \hookrightarrow & B^{n+1}\times B\times B^1 \\ & \overset{\varphi}{\searrow} & \downarrow pr_{2,3} \\ & & B\times B^1 \end{array}$$

family of hypersurfaces

$X_{s,\lambda} = \{ F(x,s) = \lambda \} = \varphi^{-1}(s,\lambda)$

$\mathrm{discr} = \{(s,\lambda) \mid X_{s,\lambda} \text{ is singular}\}$

If $U \subset B\times B^1 \setminus \mathrm{discr.}$ is open contractible then

$H^n(\varphi^{-1}(U);\mathbb{C}) \xrightarrow[\text{restriction}]{\cong} H^n(X_{s_0,\lambda_0};\mathbb{C})$ for any $(s_0,\lambda_0)\in U$

$\Rightarrow$ we can construct a vector bundle $\mathbb{H}' \longrightarrow (B\times B^1 \setminus \mathrm{discr.})$
w/ a flat connection – Gauss-Manin connection

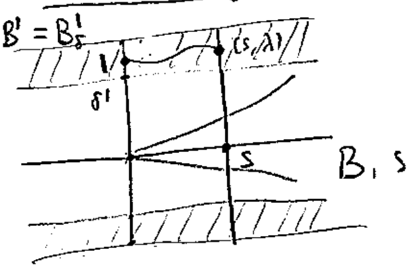
called vanishing cohomology bundle $\mathbb{H}'_{s,\lambda} = H^n(X_{s,\lambda};\mathbb{C})$.

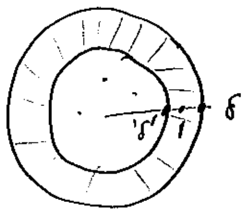
$\mathbb{H}'|_U := U \times H^n(\varphi^{-1}(U);\mathbb{C})$.

Elementary sections: Move this to page 9!

fix $\delta' < \delta$ s.t. discr. $\subset \{|\lambda|<\delta'\} \subset B\times B^1_\delta$

assume $\delta' < 1 < \delta$





$H = H^n(X_{0,1};\mathbb{C})$, M : monodromy around the discrim.

$M = M_s M_u$, $M_u = \exp(-2\pi\sqrt{-1}\,N)$, N : nilpotent operator

M_s: diag., M_u: unipotent

$H_\alpha = \{A \in H : M_s A = e^{-2\pi\sqrt{-1}\alpha} A\}$, $-1<\alpha\le 0$

For $A\in H_\alpha$, $\lambda^{\alpha+N} A_{s,\lambda} =: es_A(s,\lambda)$

$\lambda^\alpha \cdot e^{\log\lambda \cdot N}$

Geometric sections:

If $\omega \in \varphi_* \Omega^{n+1}_{X/B}$ then we can define

$$s_i[\omega](s,\lambda) \in H^n(X_{s,\lambda};\mathbb{C}) \quad , \quad i = 0, -1$$

as follows: 1) if $(s,\lambda) \notin$ discr. then $d_xF \neq 0 \quad \forall x \in X_{s,\lambda}$

$\Rightarrow$ we can find a form $\eta \in \Omega^n_{X/B}(U)$, U some neighb. of $X_{s,\lambda}$

s.t. $\omega = dF \wedge \eta$,

$$S_0[\omega](s,\lambda) = \eta|_{X_{s,\lambda}}$$

we also write $s_0[\omega] = \int \frac{\omega}{dF}$

2) Since $H^{n+1}(X_{s,\lambda};\mathbb{C}) = 0$ and $X_{s,\lambda}$ is a Stein manifold

we can find $\eta \in \Omega^n_{X/B}$ s.t. $\omega = d\eta$

$$S_{-1}[\omega](s,\lambda) = \eta|_{X_{s,\lambda}} .$$

$\nabla_{\frac{\partial}{\partial\lambda}} s[\eta \wedge dF] = s[\omega]$

Let $\omega_1, \dots, \omega_\mu \in \varphi_* \Omega^{n+1}_{X/B}(B \times B^1) = \Omega^{n+1}_{X/B}(X)$;

$\boxed{\Gamma(B^1 \setminus \{0\}, \mathbb{H}) \ \cup\!\!\mid \ \text{geometric sections are } \partial_\lambda^{-1}\text{-module as } \mathcal{O}_{B^1}\text{-module}}$

$\delta_1, \dots, \delta_\mu \in H_n(X_{0,1};\mathbb{C})$

$$\det{}^2(s,\lambda) = \left(\det\left(\langle S_0[\omega_i], \delta_j(s,\lambda)\rangle\right)\right)^2 \quad \text{: holomorphic function on } B \times B^1$$

$$= g(s,\lambda)\cdot(h(s,\lambda))^{n-1}$$

Def 3. The forms $(\omega_1, \dots, \omega_\mu)$ form a <u>holomorphic trivialization</u> if $\forall\ \omega \in \Omega^{n+1}_{X/B}(X)$ $\exists!$ holomorphic functions $P_j \in \mathcal{O}_{B\times B^1_\delta, 0}$

s.t.
$$S_0[\omega] = \sum_{j=1}^{\mu} P_j(s,\lambda)\, S_0[\omega_j] . //$$

<u>Theorem 2.</u> [Varchenko] The forms $(\omega_1, \dots, \omega_\mu)$ form a holomorphic trivialization if and only if

$$\det{}^2(s,\lambda) = g(s,\lambda)\,(h(s,\lambda))^{n-1} \quad , \quad g(0,0) \neq 0$$

where $h(s,\lambda) = 0$ is the equation of the discriminant (discr.)

$\det^2(0,\lambda)$ has a zero of order $\mu\cdot(n-1)$

$$\overset{\shortparallel}{\text{const.}}\ \lambda^{\mu(n-1)} + \text{h.o.t.}$$

Let us assume that $\omega_1, \omega_2, \dots, \omega_\mu \in \Omega^{n+1}_{X/B}(X)$ are holomorphic forms satisfying

(1) ω_1 is a volume form

(2) $K_F^{(p)}\left(\int e^{F/z}\omega_i, \int e^{F/z}\omega_j \right)\Big|_{s=0} = \delta_{i+j, \mu+1} \cdot \delta_{p,0}$

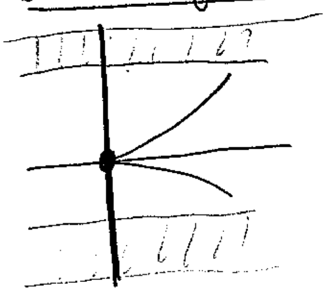
(3) $\nabla_\xi \nabla_{\frac{\partial}{\partial\lambda}}^{-1} [\omega_i] \in \bigoplus_{j=1}^{\mu} \mathcal{O}_B \cdot [\omega_j]$, $\forall \xi \in T_B$ and $\forall i = 1, 2, \dots, \mu$

(4) $\lambda \cdot [\omega_i] \in \bigoplus_{j=1}^{\mu} \mathcal{O}_B [\omega_j] + (\alpha_i + 1) \nabla_{\frac{\partial}{\partial\lambda}}^{-1} [\omega_i]$

where $\alpha_1, \dots, \alpha_\mu$ are some constants; then ω_1 is a <u>primitive form</u>.

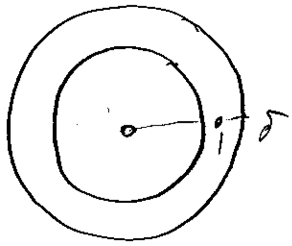
This follows from Taylor's formula + compatibility of K_F w/ the Gauss-Manin connection.

6. Hodge theory



restrict to $s = 0$

$$\begin{array}{c} X_0 \hookrightarrow B^{n+1} \times B^1 \\ \searrow^{f} \quad \downarrow pr_2 \\ B^1 \end{array}$$



$B^{n+1} \xrightarrow{f} B^1$

$X_{0,\lambda} := f^{-1}(\lambda)$

$$F_p H = \bigoplus_{-1<\alpha\le 0} F_p H_\alpha, \quad H = \mathbb{H}_{0,1} = H^n(X_{0,1}; \mathbb{C})$$

If $\omega \in \Omega^{n+1}_{X_0}$ then $s_0[\omega](0,\lambda) = \sum_{i=1}^{\mu} P_i(\lambda)\, es_{A_i}(0,\lambda)$

where $\{A_i\}_{i=1}^{\mu}$ is an eigen-basis of M_s, and $P_i(\lambda) \in \mathbb{C}\{\{\lambda\}\}$ conv. Laurent series

$$F_p H_\alpha = \{ A \in H_\alpha \mid s_0[\omega] = \lambda^p es_A(0,\lambda) + \text{h.o.t.} \text{ for some } \omega \in \Omega^{n+1}_{B^{n+1}} \}$$

Varchenko

Steenbrink

$$\nabla_{\frac{\partial}{\partial\lambda}}^{p} s_0[\omega] = es_A(0,\lambda) + \text{h.o.t.}$$

$$0 = F_{-1} \subset F_0 \subset \cdots \subset F_n = H$$

$$K_f(s[\omega_1], s[\omega_2]) := K_f(\omega_1, \omega_2), \quad \forall \omega_1, \omega_2 \in \varphi_* \Omega^{n+1}_{X_0}$$

$$\cancel{K_f(s[\omega_1], s[\omega_2]) \neq K_f(\nabla_{\frac{\partial}{\partial\lambda}} s[\omega_1], s[\omega_2])}$$

$$K_f(\underbrace{\nabla^{-1}_{\frac{\partial}{\partial\lambda}} s[\omega_1]}_{s_1}, \underbrace{s[\omega_2]}_{s_2}) = -z\, K_f(s[\omega_1], s[\omega_2])$$

$$K_f(s_1, s_2) = -z\, K_f(\nabla_{\frac{\partial}{\partial\lambda}} s_1, s_2) \qquad (*)$$

we can uniquely extend K_f to a pairing on V

$$K_f : V \otimes V \longrightarrow \mathbb{C}[z]z$$

s.t. $(*)$ holds.

$$K_f(\lambda^{\alpha+N}A, \lambda^{\beta+N}B) = \begin{cases} S(A,B)\frac{z}{2\pi\sqrt{-1}} & \text{if } \alpha+\beta=-1 \Rightarrow A,B \in H_{\neq 0} \\ S(A,B)\frac{z^2}{(2\pi\sqrt{-1})^2} & \text{if } \alpha=\beta=0 \text{ (hence } A,B\in H_0) \end{cases}$$

$-1 < \alpha, \beta \leq 0$

where $S : H_{\mathbb{Q}} \otimes H_{\mathbb{Q}} \to \mathbb{Q}$ is $(-1)^n$-symmetric on $H_{\neq 0}$ and $(-1)^{n+1}$-symmetric on $H_{=0}$, and it is M-invariant.

<u>Lemma</u> 4 [Hertling]

$$S(A,B) := \begin{cases} \text{sign } A \circ B & \text{for } A,B \in H_{\neq 0}; \\ \text{sign } \langle A, \mathrm{Var} \circ \frac{N}{e^{-2\pi\sqrt{-1}N}-1} B \rangle & \text{for } A,B\in H_0. \end{cases}$$

($A \circ B$: intersection pairing)

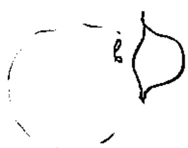
$$H^n(X_{0,1}; \mathbb{C}) \cong H_n(X_{0,1}, \partial X_{0,1}; \mathbb{C})$$

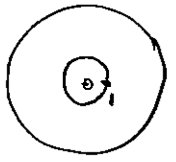
$$\mathrm{Var}\, b = \Gamma_* b - b$$

X_0

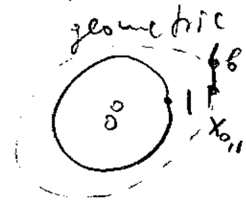
$f \downarrow$

B^1





$\Gamma_* : X_{0,1} \to X_{0,1}$ geometric monodromy fixing $\partial X_{0,1}$



$$\partial X_0 \subset \partial B^{n+1} \times B'$$

$f \searrow \quad \downarrow$ trivial fibration

B^1 because B^1 is contractible

$X_{0,\lambda} \pitchfork \partial B^{n+1} \Rightarrow X_{0,\lambda} \cap \partial B^{n+1}$

$H_{\mathbb{R}} = H^n(X_{0,1};\mathbb{R}) \subset H$, $F^pH := F_{n-p}H$ decreasing filtration

real structure

a non-degen. M-invariant pairing $S : H \otimes H \to \mathbb{C}$

N : nilpotent operator $\quad M_u = \exp(-2\pi\sqrt{-1}\cdot N)$.

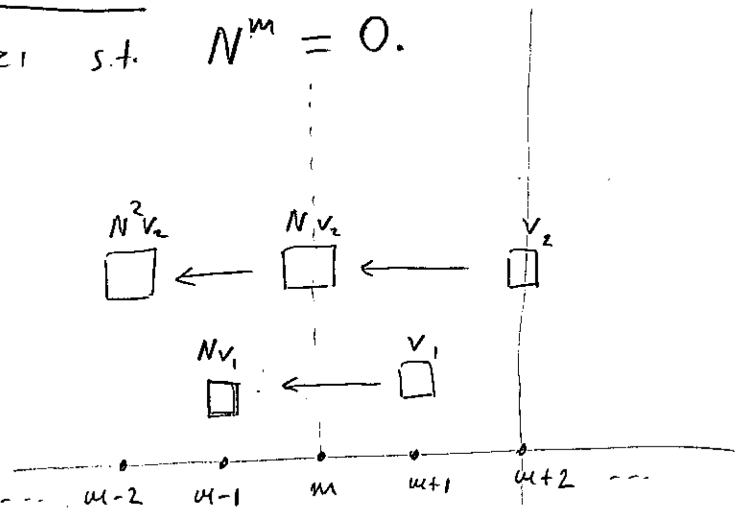
<u>Theorem 3</u>. [Steenbrink, Varchenko, Hertling] Let $W_\bullet H$ be the weight filtr. of N

$\Rightarrow (H_{\neq 0}, H_{\mathbb{R},\neq 0}, F^p, S, W_\bullet)$ is a mixed Hodge structure of weight n

$(H_0, H_{\mathbb{R},0}, F^p, S, W_\bullet)$. — " — weight $n+1$

<u>Weight filtration</u>: $S(NA, B) + S(A, NB) = 0$, N : nilpotent

$m \in \mathbb{Z}_{\geq 1}$ s.t. $N^m = 0$.



Given a Jordan block $v, Nv, \ldots, N^kv$ ($v \notin \operatorname{Im} N$, $N^{k+1}v = 0$) we assign weights according to the picture on the left

$0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2m} = H$

$W_i H = \operatorname{span}_{\mathbb{C}}\{\text{vectors of weight} \leq i\}$,

$Gr_{m+i}W$ has a non-degen. pairing

$Gr_i W = W_i / W_{i-1}$; $S_i(v_1, v_2) := S(v_1, N^i v_2)$

<u>Def. 4</u> <u>mixed Hodge structure of weight m means</u>:

(1) $Gr_k W = F^p Gr_k W \oplus \overline{F^{k+1-p} Gr_k W}$, $\forall p$ and $\forall k$,

(2) $N(F^p) \subset F^{p-1}$,

(3) $S(F^p, F^{m+1-p}) = 0 \quad \forall p$,

(4) some additional property of $P_{m+\ell} = \operatorname{Ker}\left(Gr_{m+\ell} W \xrightarrow{N^\ell} Gr_{m-\ell} W\right)$.

<u>Corollary 1. [Deligne]</u> <u>Opposite filtration</u>

$\exists$ an increasing M-invariant filtration $U_p H$, s.t.

(i) $H = \bigoplus_p F^p \cap U_p$,

(ii) $N(U_p) \subset U_{p-1}$,

(iii) $S(U_p H_{\neq 0}, U_{n-1-p} H_{\neq 0}) = S(U_p H_0, U_{n-p} H_0) = 0 \quad \forall p$.

Fix a basis $\{A_i\}_{i=1}^{\mu}$ of H s.t. $A_i \in F^{p_i} H_{\alpha_i} \cap U_{p_i} H_{\alpha_i}$

$$\sigma_i(\lambda) = \nabla_\lambda^{-n+p_i}(\lambda^{\alpha_i + N} A_i) \sim \lambda^{\alpha_i - p_i + n + N} A_i \qquad -1 < \alpha_i \le 0$$

<u>Step 1.</u> Choose holom. forms $\tilde{\omega}_1, \dots, \tilde{\omega}_\mu \in \Omega^{n+1}_{X_0}(X_0)$

s.t. $$s[\tilde{\omega}_i](\lambda) = \sigma_i(\lambda) + \sum_{\substack{j,p:\\ p>0,\ a_j > a_i + p}} c_{j,i,p} \nabla_\lambda^{p} \sigma_j(\lambda)$$

$$\alpha_j + p_j > \alpha_i + p_i + p$$

Condition (3) in Def 4 and (iii) in Corollary 1, imply

$$S(F^{p'} \cap U_{p'}, F^{p''} \cap U_{p''}) = 0$$

for $p' + p'' \neq m = \begin{cases} n & \text{on } H_{\neq 0} \\ n+1 & \text{on } H_{=0} \end{cases}$

The numbers $a_i = \alpha_i - p_i + n$ are called spectral numbers

if ordered $a_1 \le \dots \le a_\mu \Rightarrow a_i + a_{\mu+1-i} = n-1 \Rightarrow$ $(\tilde{\omega}_1, \dots, \tilde{\omega}_\mu)$ give a holom. trivialization

$\Rightarrow \quad K_f^{(p)}(\sigma_i, \sigma_j) = 0$ for $\forall p \ge 1 \quad \forall\, i, j$

$\Rightarrow \quad K_f^{(p)}(\tilde{\omega}_i, \tilde{\omega}_j) = 0$ for $p \ge 1$

$$\tilde{\omega}_i = \tilde{g}_i(x)\, d^{n+1}x \in \Omega^{n+1}_{X/B}$$

$$s[\tilde{\omega}_i](s,\lambda) = \sum_{j=1}^{\mu} B_{ji}(s,\lambda)\ \sigma_j(\lambda)$$

holomorphic in $\delta' < |\lambda| < \delta$

$$B_g(s,\lambda) = L(s,\lambda)\, U(s,\lambda)^{-1}$$

L: $1 + O(\frac{1}{\lambda})$ U: holom. in $|\lambda| < \delta$

Birkhof factoriz. (it exists for $s=0$ so exists for s suff. small)

Put $\omega_i := \sum_j \tilde{\omega}_j\ U_{ji}(s, F(x))$

then a) $s[\omega_i] = \sum_{j=1}^{\mu} L_{ji}(s,\lambda)\ \sigma_j(\lambda)$

b) $(\omega_1, -, \omega_\mu)$ still give a holom. trivialization

From a) and b) one obtains

$$\nabla_\xi \nabla_\lambda^{-1} s[\omega_i] = \sum_{j=1}^{\mu} c_{ji}(s,\lambda)\ s[\omega_j]$$

where $c_{ji}(s,\lambda)$ are holomorphic $\forall \lambda \in \mathbb{C}$ and bounded at $\lambda = \infty$ $\Rightarrow$ c_{ji} are independent of λ.