Horizon instability of an extreme Reissner-Nordström black hole

Norihiro Tanahashi (Kavli IPMU)

in collaboration with

James Lucietti, Keiju Murata, Harvey S. Reall

based on **arXiv:1212.2557**

(In)stability of Black holes

– Classical stability of 4D Black holes
  – Stable (mostly)
  – Power-law decay of perturbations

– ***Extreme*** black holes in 4D

  à Novel instability on the horizon

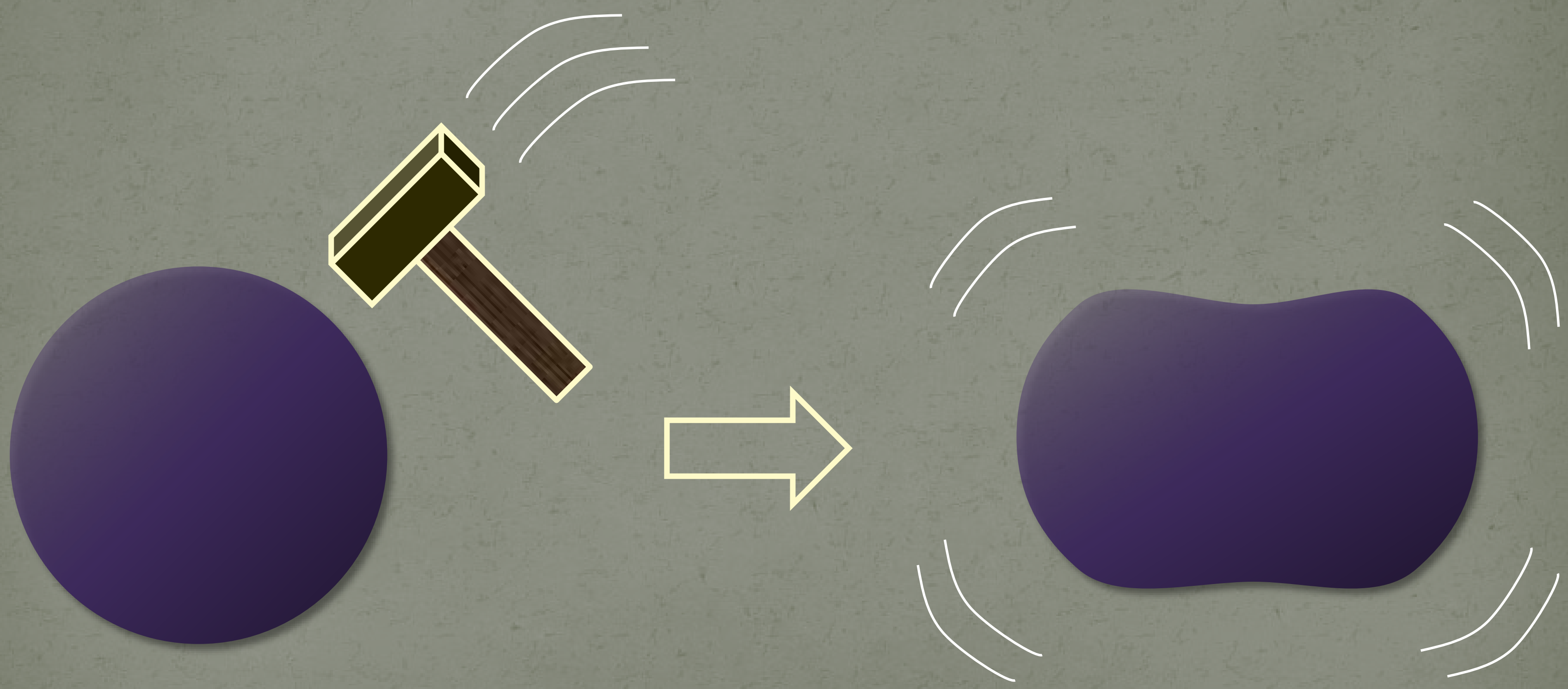
  – Scalar fields
  – EM & Gravitational perturbations

Contents

1. 1957 – 2010
   BH Stability & Price's law
2. 2011
   Instability of extreme horizons
3. 2012
   Generalizations
4. 2013
   Backreaction

1957 – 2010: BH Stability & Price's law

- BH Stability:



[Regge & Wheeler "Stability of a Schwarzschild singularity" 1957]

1957 – 2010: BH Stability & Price's law

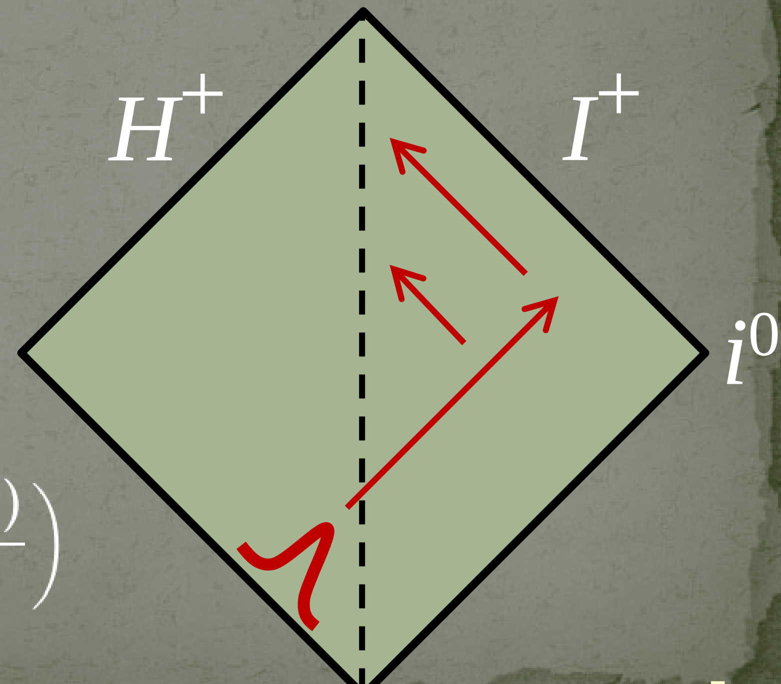
- Price's law (1972):
  - Scalar field on BH decays by power law:

$$\psi_\ell (t, r = r_0) \mu \; t^{-(2\ell+3)}$$

Due to backscattering

$$-\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial r_*^2} - V(r)\phi = 0$$

$$V(r) = \left(1 - \frac{2M}{r}\right)\left(\frac{2M}{r^3} + \frac{\ell(\ell+1)}{r^2}\right)$$



1957 – 2010: BH Stability & Price's law

- Price's law (1972):
  - Scalar field on BH decays by power law:

$$\psi_\ell (t, r = r_0)\ \mu\ t^{-(2\ell+3)}$$

  - By heuristic arguments

- Confirmations
- Generalizations (Sch. → Kerr, RN, …)
- Also on H^+ and I^+ [Gundlach+ 1993]
- Quasi-normal modes
- . . .

1957 – 2010: BH Stability & Price's law

– Rigorous results:

Uniform boundedness [Wald 1979, Kay & Wald 1987]

Regular initial data $\psi\,(t_0, r)$ on Schwarzschild

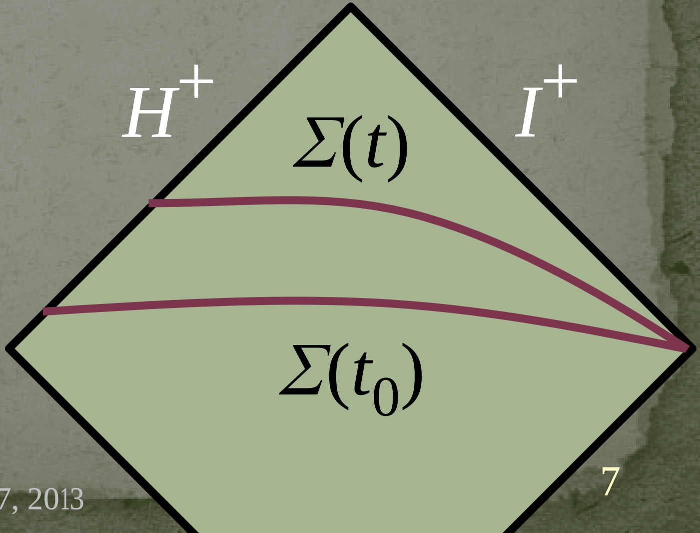
à $|\,\psi(t,r)\,| < C\,[\psi(t_0)\,]$

everywhere including H^+

ü Energy estimate

$$E[\Sigma(t)] \leq E[\Sigma(t_0)]$$

$$\approx \int_{\Sigma(t)} (\partial_t \psi)^2 + (\partial_r \psi)^2 + (\partial_\Omega \psi)^2$$



1957 – 2010: BH Stability & Price's law

– Rigorous results:

Decay laws:

– Scalar field on stationary asym. flat spacetime

à $|\psi(t,r)| < C[\psi(t_0)]$ ´ t^{-3}

[Tataru 2010]

– Spherically sym. Scalar on Schwarzschild

à $|\psi(t)| + |\partial_t \psi(t)| < C[\psi(t_0)]$ ´ t^{-3}

including ***backreaction***

[Dafermos & Rodnianski 2004]

– Many others…

2011: Instability of extreme horizons

– Non-extreme BHs are stable

– ***Extreme*** BH?

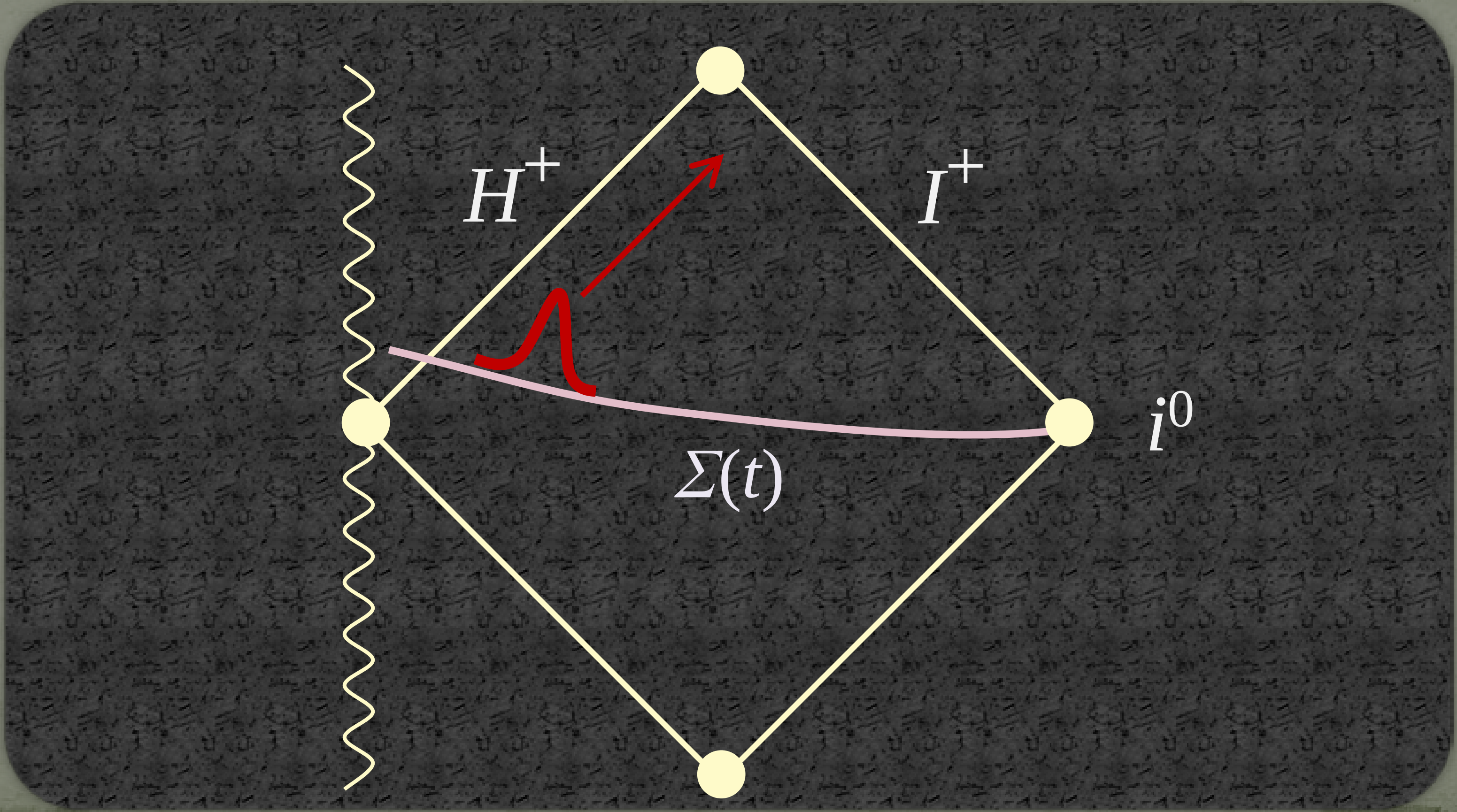
  - $\boldsymbol{\kappa = 0}$ → New phenomena on the horizon

– Aretakis' results:

  - $\psi(t,r)$ and its derivatives ***decay around BH***
  - ***Non-decay & blow-up on horizon***

2011: Instability of extreme horizons

- Aretakis' argument:
  - Massless scalar on extreme Reissner-Nordström



2011: Instability of extreme horizons

– Aretakis' argument: [Aretakis 2010, 2011]

– Massless scalar on extreme Reissner-Nordström

For $\psi = \psi(v, r)$,

– Pointwise decay:

– $| \psi(v,r) | < C[\psi(v_0)]$ ´ $v^{-3/5}$

– Non-decay & Blow-up on horizon:

– $| \partial_r \psi(v,r) | \rightarrow H [\psi(v_0)]$

– $| \partial_r^{\boldsymbol{k}} \psi(v,r) | > H [\psi(v_0)]$ ´ $\boldsymbol{v}^{\boldsymbol{k\text{-}1}}$ ($k \geq 2$)

2011: Instability of extreme horizons

– Aretakis' argument: [Aretakis 2010, 2011]
  – Massless scalar on extreme Reissner-Nordström

For $\psi = \sum_{\ell' \geq \ell}^{\infty} \psi_{\ell'} Y_{\ell'}$,

– Pointwise decay:
  – $| \psi(v,r) | < C[\psi(v_0)] \times v^{-3/5}$
  – $| \partial_r^{\,n \leq \ell} \psi(v,r) | < C[\psi(v_0)] \times v^{-1/4}$

– Non-decay & Blow-up on horizon:
  – $| \partial_r^{\,\ell+1} \psi(v,r) | \to H_\ell[\psi(v_0)]$
  – $| \partial_r^{\,\ell+k} \psi(v,r) | > H_\ell[\psi(v_0)] \times v^{k-1}$ $(k \geq 2)$

2011: Instability of extreme horizons

– Aretakis' argument:
  - ü $ Conserved quantities on extreme horizon

$$ds^2 = -\left(1 - \frac{M}{r}\right)^2 dv^2 + 2dvdr + r^2 d\Omega^2$$

$$\Box\psi = 0 \quad \text{w/} \quad \psi(v, r, \Omega) = \sum_{\ell=0}^{\infty} \psi_\ell(v, r) Y_\ell(\Omega)$$

è

$$2r\partial_v\partial_r(r\psi_\ell) + \partial_r(\Delta\partial_r\psi_\ell) - \ell(\ell+1)\psi_\ell = 0$$

$$\left(\Delta \equiv (r-M)^2\right)$$

2011: Instability of extreme horizons

– Aretakis' argument:
  - ü $ Conserved quantities on extreme horizon

$$r = M \quad \& \quad \ell = 0$$

$$\Rightarrow \quad \partial_v \partial_r (r\psi_0)\big|_{r=M} = 0$$

$$\Rightarrow \quad \partial_r (r\psi_0)\big|_{r=M} = H_0[\psi] = \text{const.}$$

$$2r\partial_v\partial_r(r\psi_\ell) + \partial_r(\Delta\partial_r\psi_\ell) - \ell(\ell+1)\psi_\ell = 0$$

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2011: Instability of extreme horizons

– Aretakis' argument:

ü $ Conserved quantities on extreme horizon

∂_r à $r = M$ & $\ell = 0$

Þ $\partial_v \partial_r^2 (r\psi_0) + \partial_r \psi_0 \big|_{r=M} = 0$

Þ $\partial_r^2 (r\psi_0) \big|_{r=M} \to -H_0 \, v$

$$2r\partial_v\partial_r(r\psi_\ell) + \partial_r(\Delta\partial_r\psi_\ell) - \ell(\ell+1)\psi_\ell = 0$$

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2011: Instability of extreme horizons

– Aretakis' argument:
  - ü $ Conserved quantities on extreme horizon

For " ℓ ,

$$\partial_r^\ell \left[r\partial_r (r\psi_\ell) \right]\Big|_{r=M} = H_\ell[\psi] = \text{const.}$$

Þ $\partial_r^{\ell+k}\psi_\ell \propto v^{k-1}$

$$2r\partial_v\partial_r(r\psi_\ell) + \partial_r(\Delta\partial_r\psi_\ell) - \ell(\ell+1)\psi_\ell = 0$$

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2011: Instability of extreme horizons

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For $$\psi = \sum_{\ell' \geq \ell}^{\infty} \psi_{\ell'} Y_{\ell'} ,$$

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  - $| \partial_r^{n \leq \ell} \psi(v,r) | < C[\psi(v_0)]$ ´ $v^{-1/4}$
- Non-decay & Blow-up on horizon:
  - $| \partial_r^{\ell+1} \psi(v,r) | \to H_\ell[\psi(v_0)]$
  - $| \partial_r^{\ell+k} \psi(v,r) | > H_\ell[\psi(v_0)]$ ´ v^{k-1} $(k \geq 2)$

2012: Generalizations

- Aretakis' argument: [Aretakis 2011, 2012]
  - $ Conserved quantities on extreme horizons
  - Massless scalar on extreme RN & Kerr

- Instability is ubiquitous
  - Gravitational perturbations on extreme Kerr [Lucietti & Reall 2012]
  - Grav. perturbations on high-D extreme horizons [Murata 2012]
  - $H_\ell = 0$ case
  - Massive scalar
  - Coupled EM & Grav. perturbations [Lucietti, Murata, Reall, NT 2012]

2012: Generalizations

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2012: Generalizations

– Main results: [Lucietti, Murata, Reall, NT 2012]

Instability for

1. $H_\ell = 0$
2. Massive scalar
3. Coupled EM & Grav. perturbations

on extreme RN background

2012: Generalizations

1. Instability for $H_\ell = 0$

   – Horizon instability µ H_ℓ

   – What if $\boldsymbol{H_\ell = 0}$?

      a. No Instability
      b. $ Instability, same growth rate
      c. $ Instability, slower growth rate

      - Confirm numerically

2012: Generalizations

1. Instability for $H_\ell = 0$

   – Horizon instability μ H_ℓ

   – What if $\boldsymbol{H_\ell = 0}$?

      a. No Instability
      b. $ Instability, same growth rate
      c. $ Instability, slower growth rate

      - Confirm numerically

2012: Generalizations

1. Instability for $H_\ell = 0$

– Numerics:

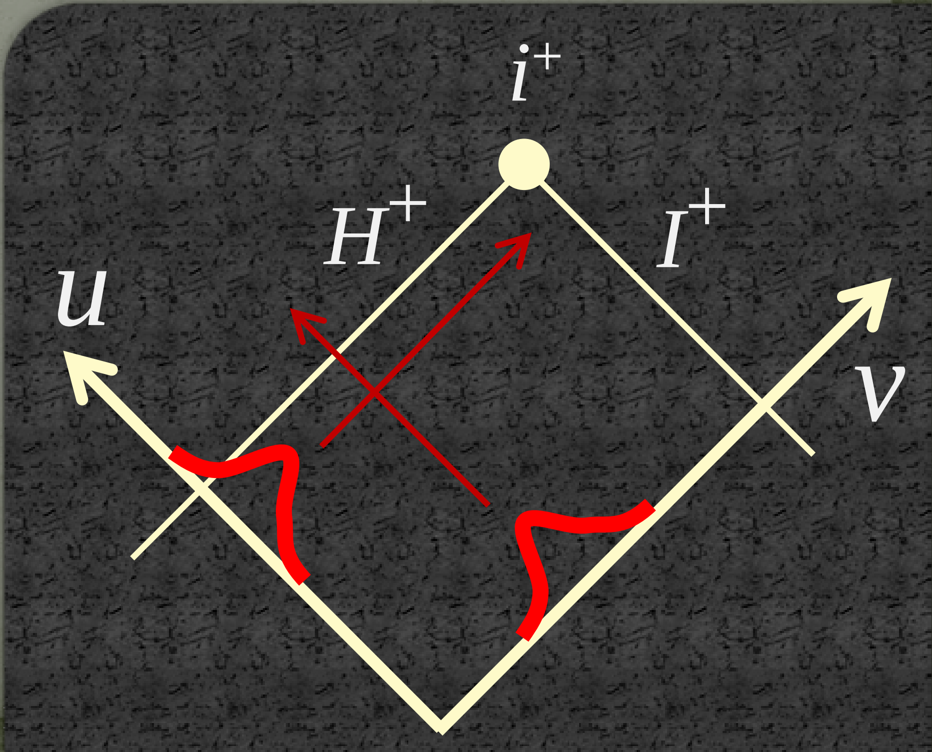
ü Massless scalar field on extreme RN

ü Double-null coordinates

$$ds^2 = -F(u,v)\,du\,dv + r^2 d\Omega^2$$

$$-\partial_u \partial_v \phi = V(u,v)\phi$$

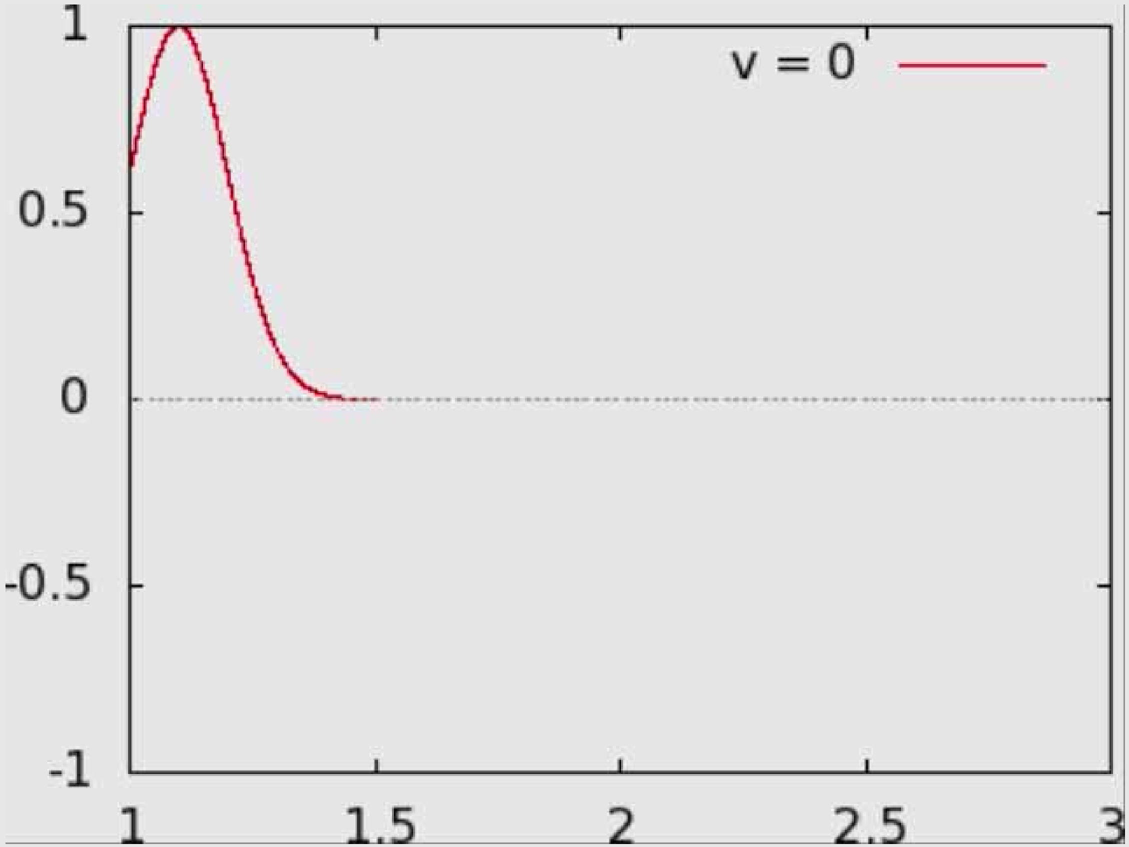
$$\left[\phi \equiv r\psi\right]$$



2012: Generalizations

– $H_{\ell=0} \neq 0$

$\phi(r)$



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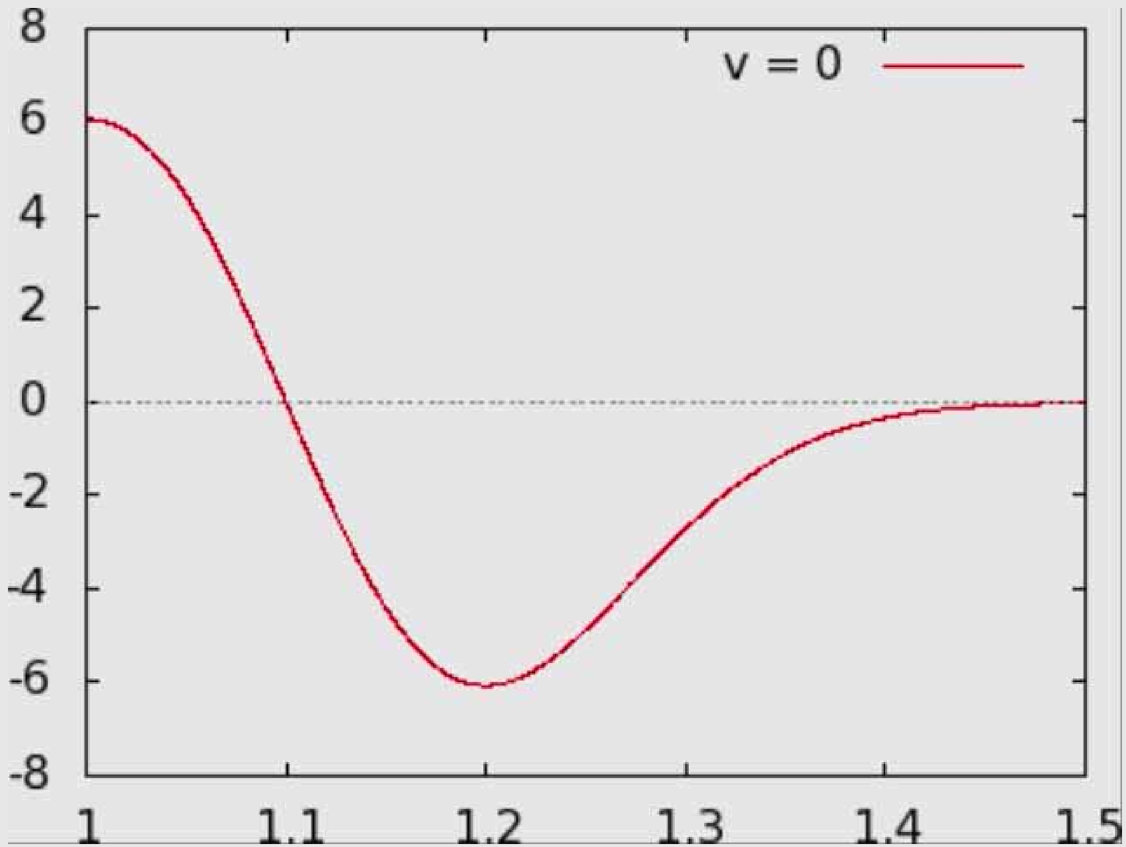
2012: Generalizations

– $H_{\ell=0} \neq 0$

$\partial_r \phi(r)$

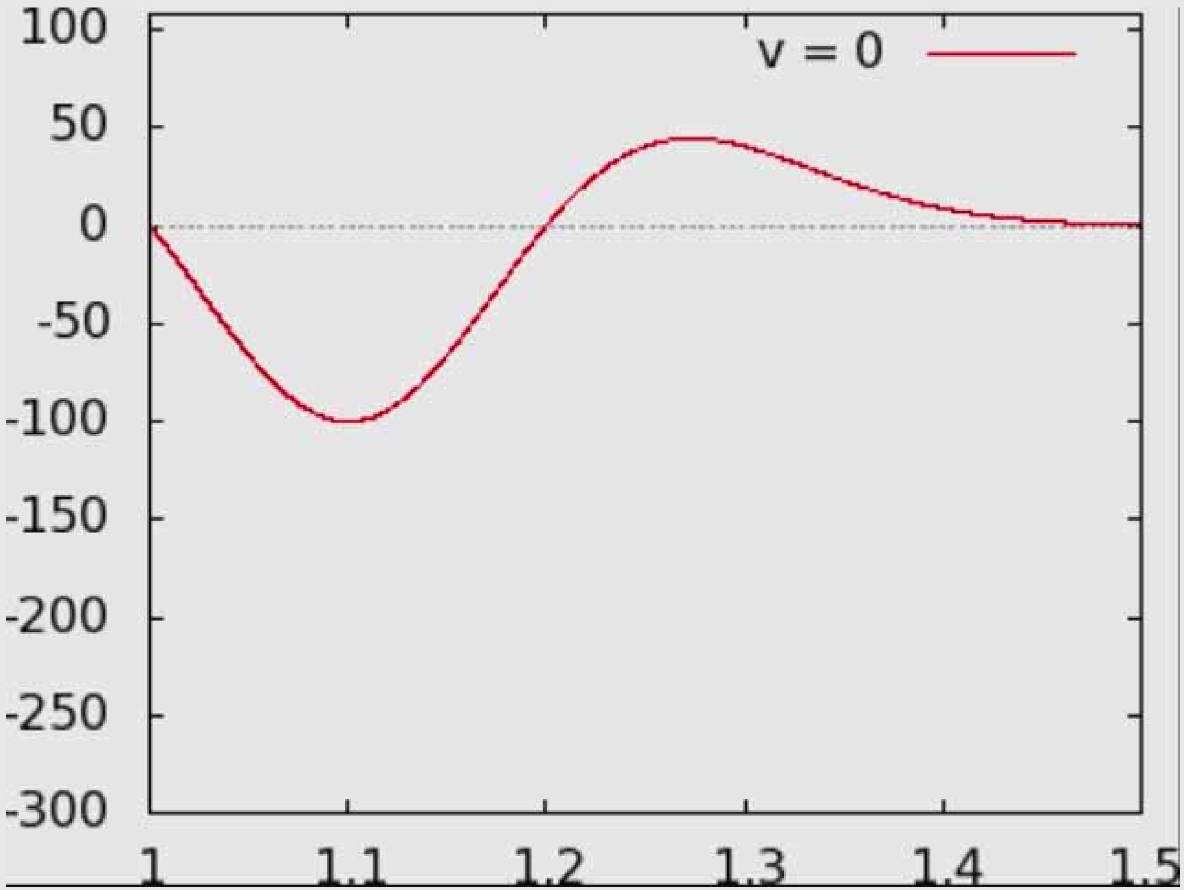


r

2012: Generalizations

– $H_{\ell=0} \neq 0$

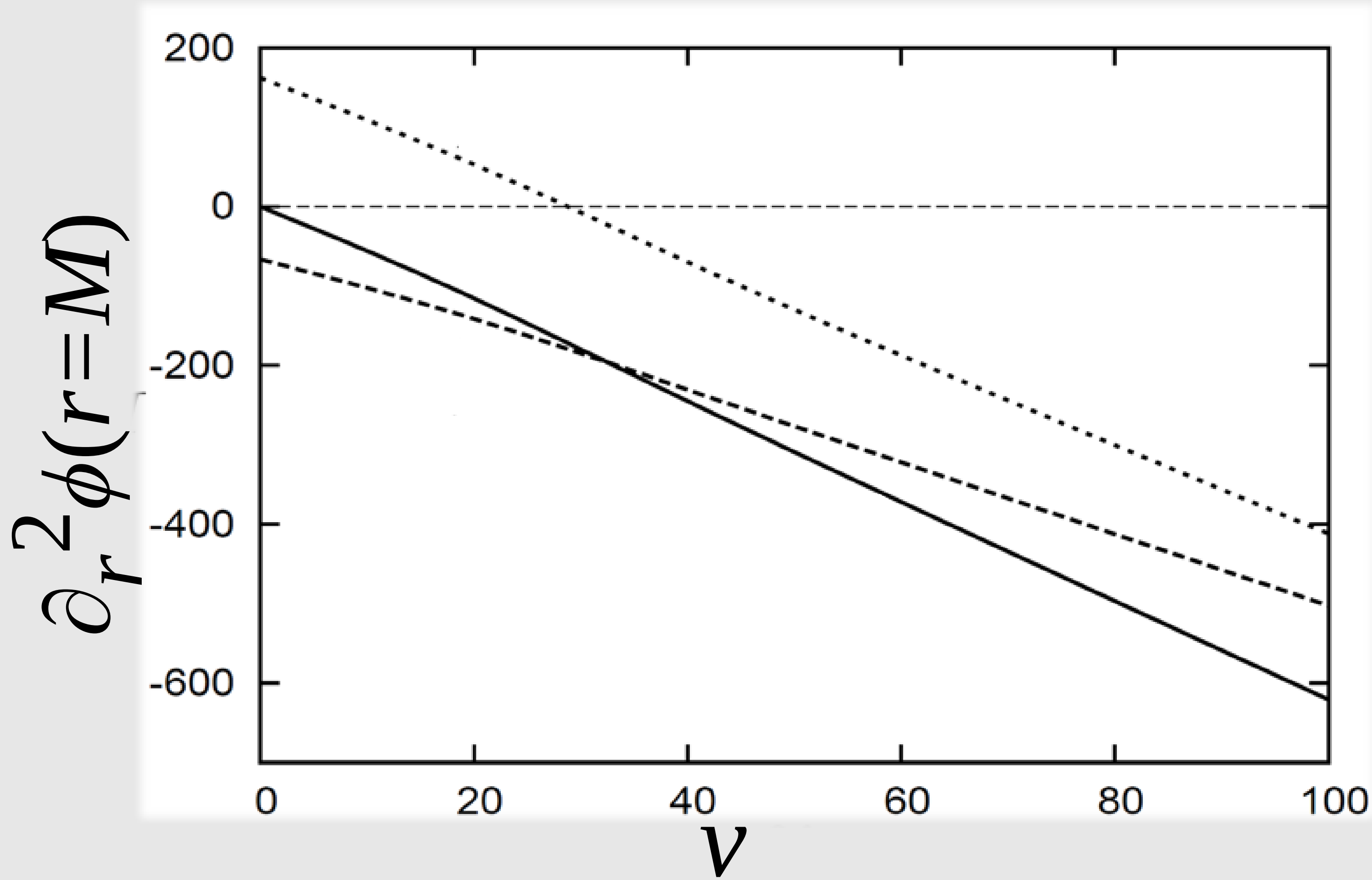
$\partial_r{}^2 \phi(r)$



r

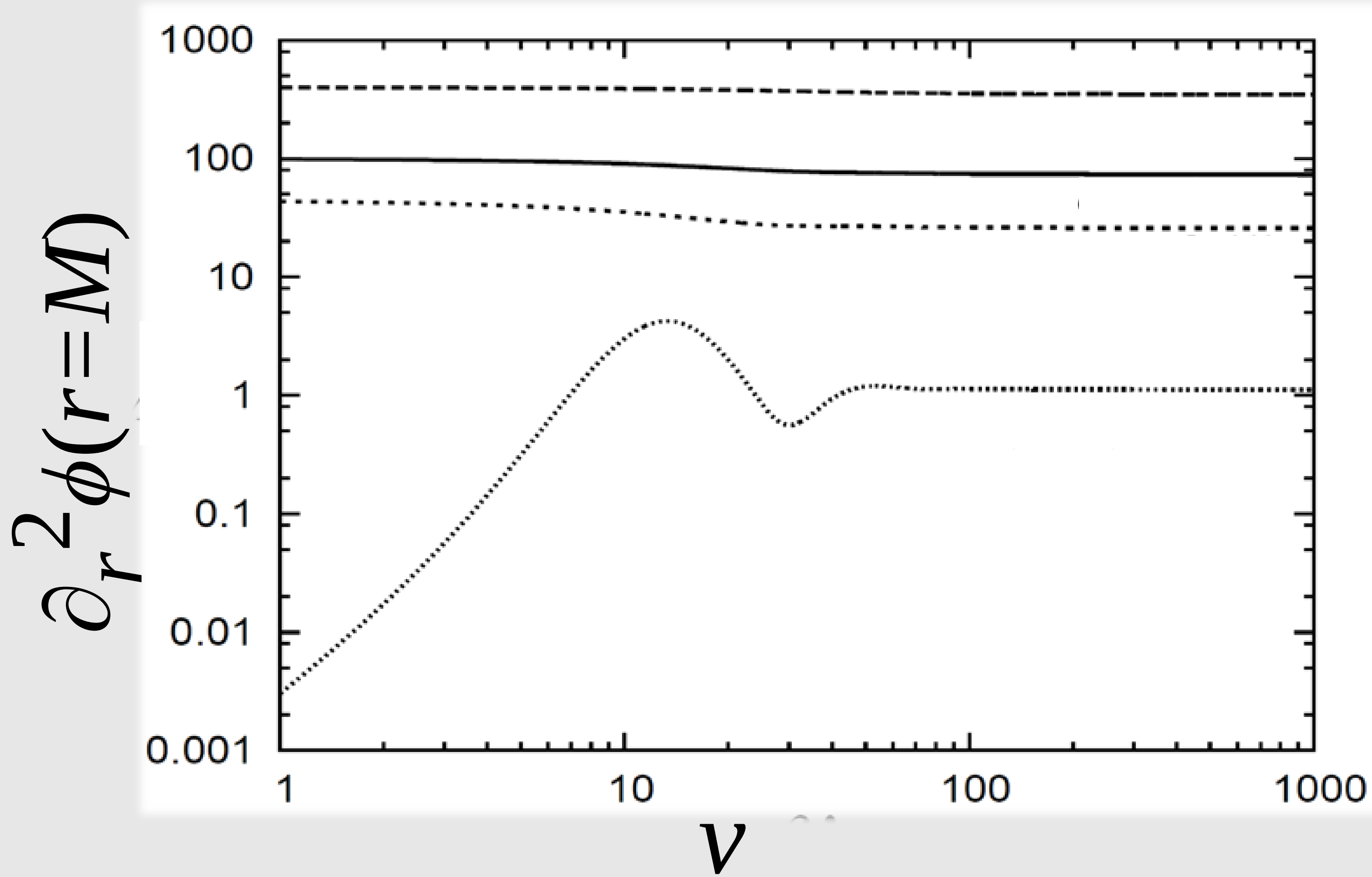
2012: Generalizations

- $H_{\ell=0} \neq 0$: $\partial_r^2 \phi_0\big|_{r=M} \to -H_0\, v$



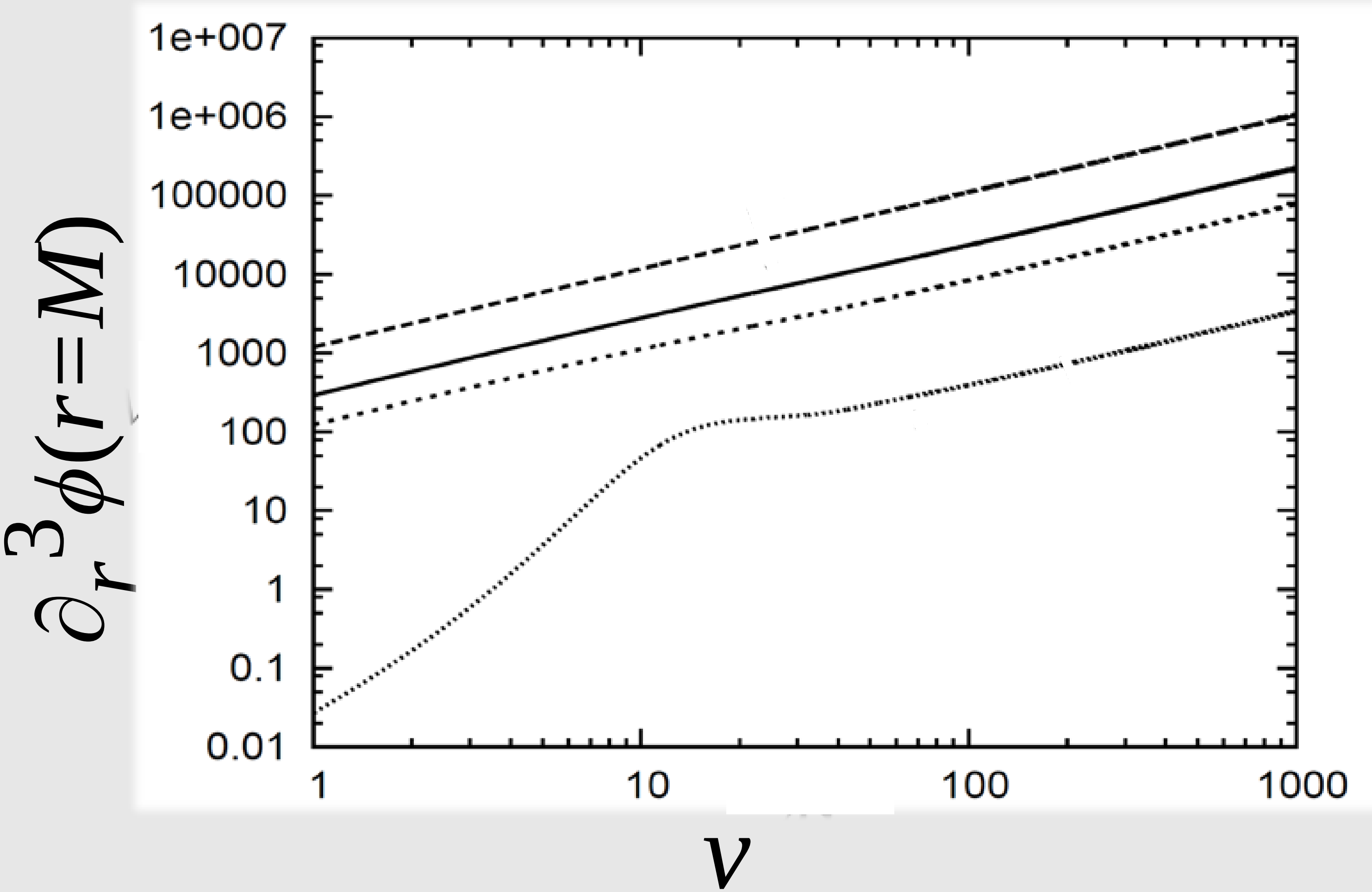
2012: Generalizations

– $\mathbf{H_{\ell=0} = 0}$: $\partial_r^2 \phi_0\big|_{r=M} \to C\, v^0$



2012: Generalizations

– $H_{\ell=0} = 0$: $\partial_r^3 \phi_0\big|_{r=M} \to C'\, v^1$



2012: Generalizations

1. Instability for $H_\ell = 0$

– Numerical results:

– $H_{\ell=0} \neq 0$:

$$\partial_r^2 \phi\big|_{r=M} \to -H_0 v$$

– $H_{\ell=0} = 0$:

$$\partial_r^2 \phi\big|_{r=M} \to C$$

$$\partial_r^3 \phi\big|_{r=M} \to C' v$$

[Match with Aretakis' proof (2012)]

2012: Generalizations

2. Instability for massive scalar

$$2r\partial_v\partial_r\phi + \partial_r\left(\Delta\partial_r\frac{\phi}{r}\right) - \left(\frac{\ell(\ell+1)}{r^2} + m^2\right)r\phi = 0$$

– Conserved quantity?

i. Set $m^2 = n(n+1)M^{-2}$ & $\ell = 0$
ii. Apply $\partial_r^n f_n(r)$, then set $r = M$

$$f_n(r) = \sum_{k=0}^{n} c_k\left(\frac{r}{M} - 1\right)^k$$

è $\partial_v\partial_r^n\left[f_n(r)\partial_r\phi\right] + \sum_{k=0}^{n} a_k\,\partial_r^k\phi\,\Big|_{r=M} = 0$

2012: Generalizations

iii. Find $c_{k=1,\ldots,n}$ to make $a_{k=1,\ldots,n} = 0$

è $H_n[\phi] = \partial_r^n \left[f_n(r) \partial_r \phi \right]\Big|_{r=M}$ conserved

i. Set $m^2 = n\,(n+1)\,M^{-2}$ & $\ell = 0$

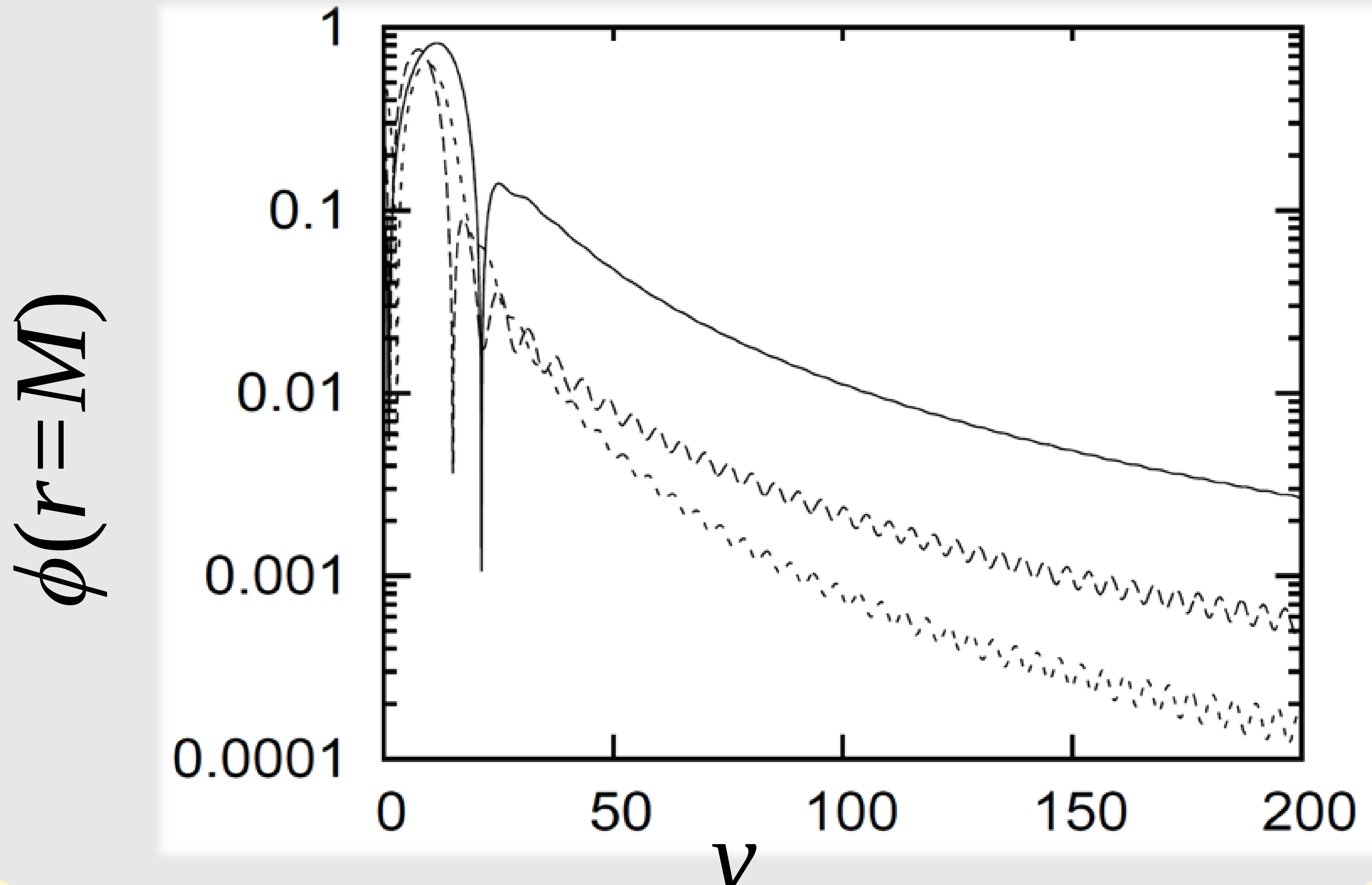
ii. Apply $\partial_r^{\,n} f_n(r)$, then set $r = M$

$$f_n(r) = \sum_{k=0}^{n} c_k \left(\frac{r}{M} - 1 \right)^k$$

è $$\partial_v \partial_r^n \left[f_n(r) \partial_r \phi \right] + \sum_{k=0}^{n} a_k \, \partial_r^k \phi \, \Big|_{r=M} = 0$$

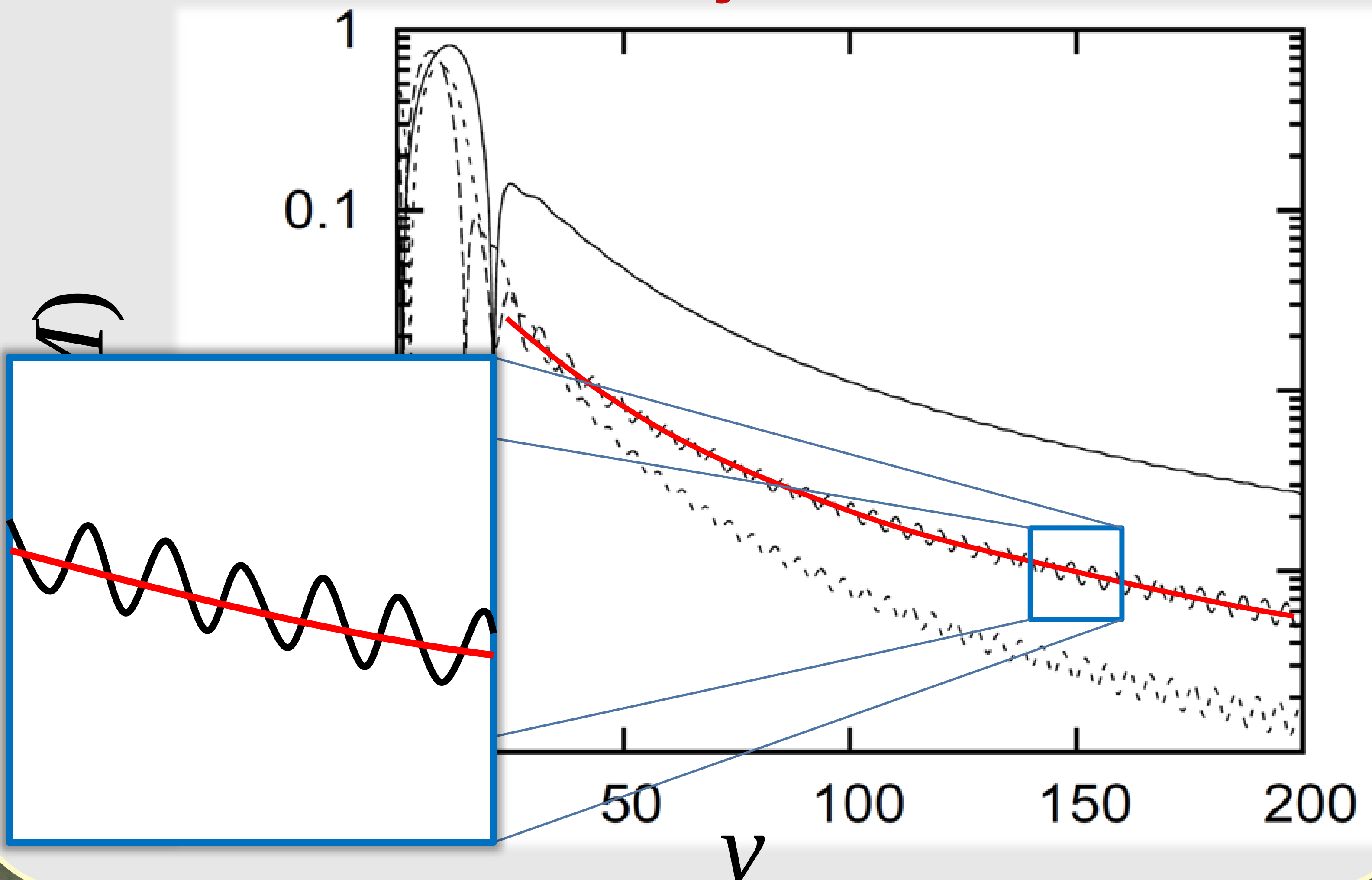
2012: Generalizations

- $m^2 = 2$



2012: Generalizations

– $m^2 = 2$ ***Decay* + *Oscillation***



2012: Generalizations

$$\phi\big|_{r=M}(v) = \phi^{\text{mean}}(v) + \phi^{\text{osci}}(v)\big|_{r=M}$$

$$\phi^{\text{mean}}(v)\big|_{r=M} \equiv T^{-1}\int_{v-T/2}^{v+T/2} \phi\big|_{r=M}(v')dv'$$

$$T = m/2\pi$$

– Damped oscillation on RN background:

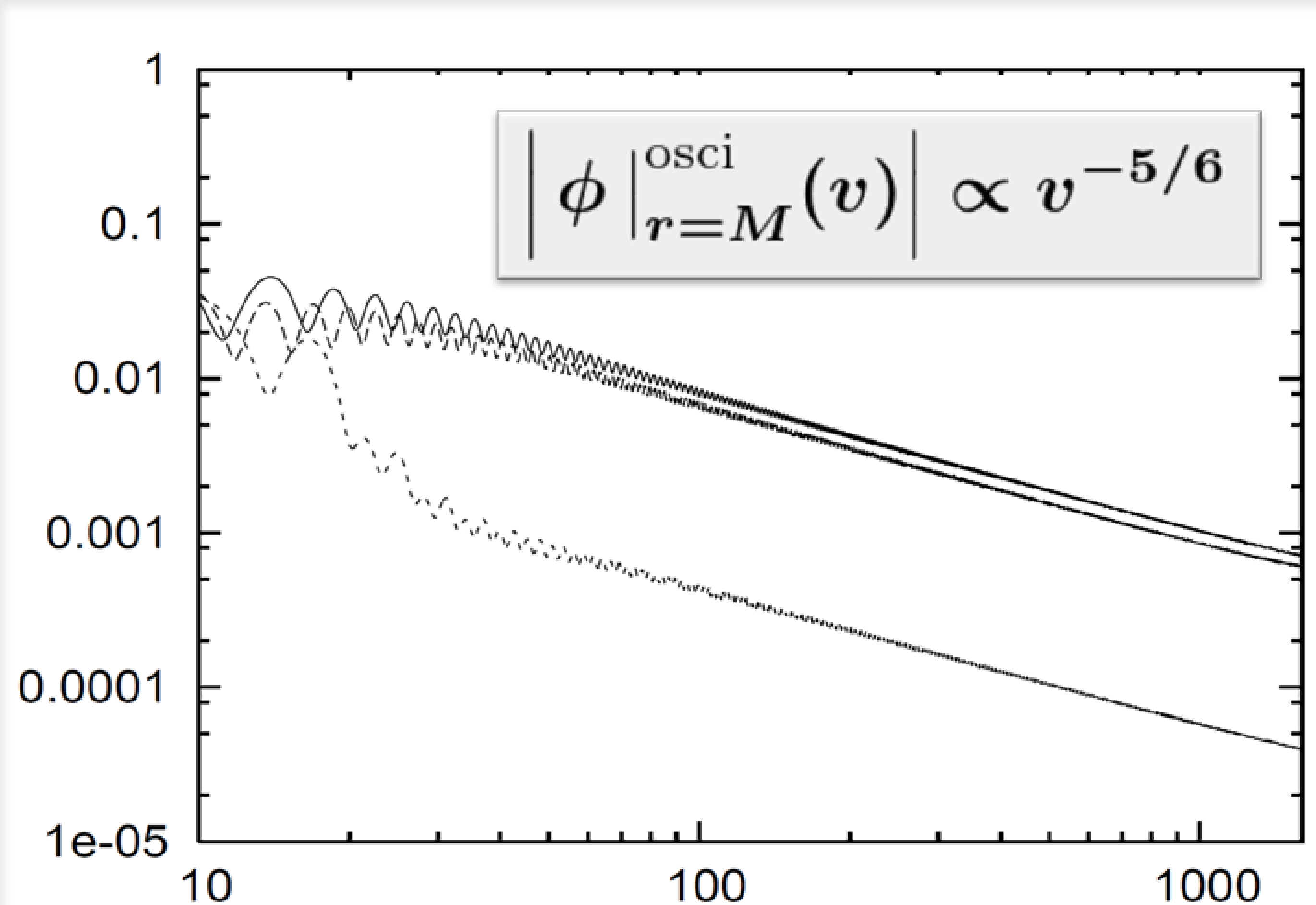
$$\left|\phi\,\right|^{\text{osci}}_{r=M}(v)\Big| \propto v^{-5/6}$$

[Koyama & Tomimatsu 2000]

2012: Generalizations

– $m^2 = 2$

$|\phi^{\rm osci}(r=M)|$



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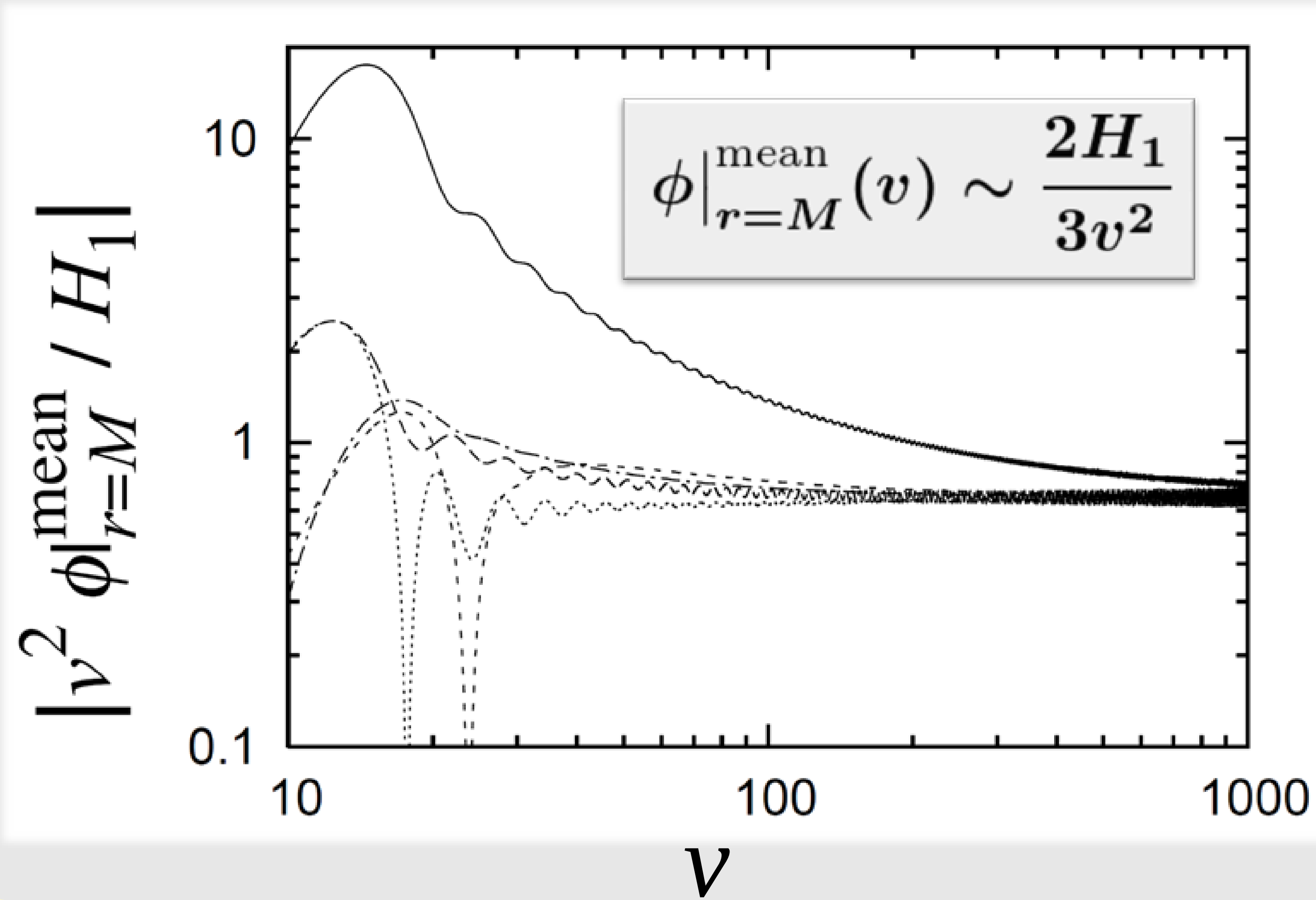
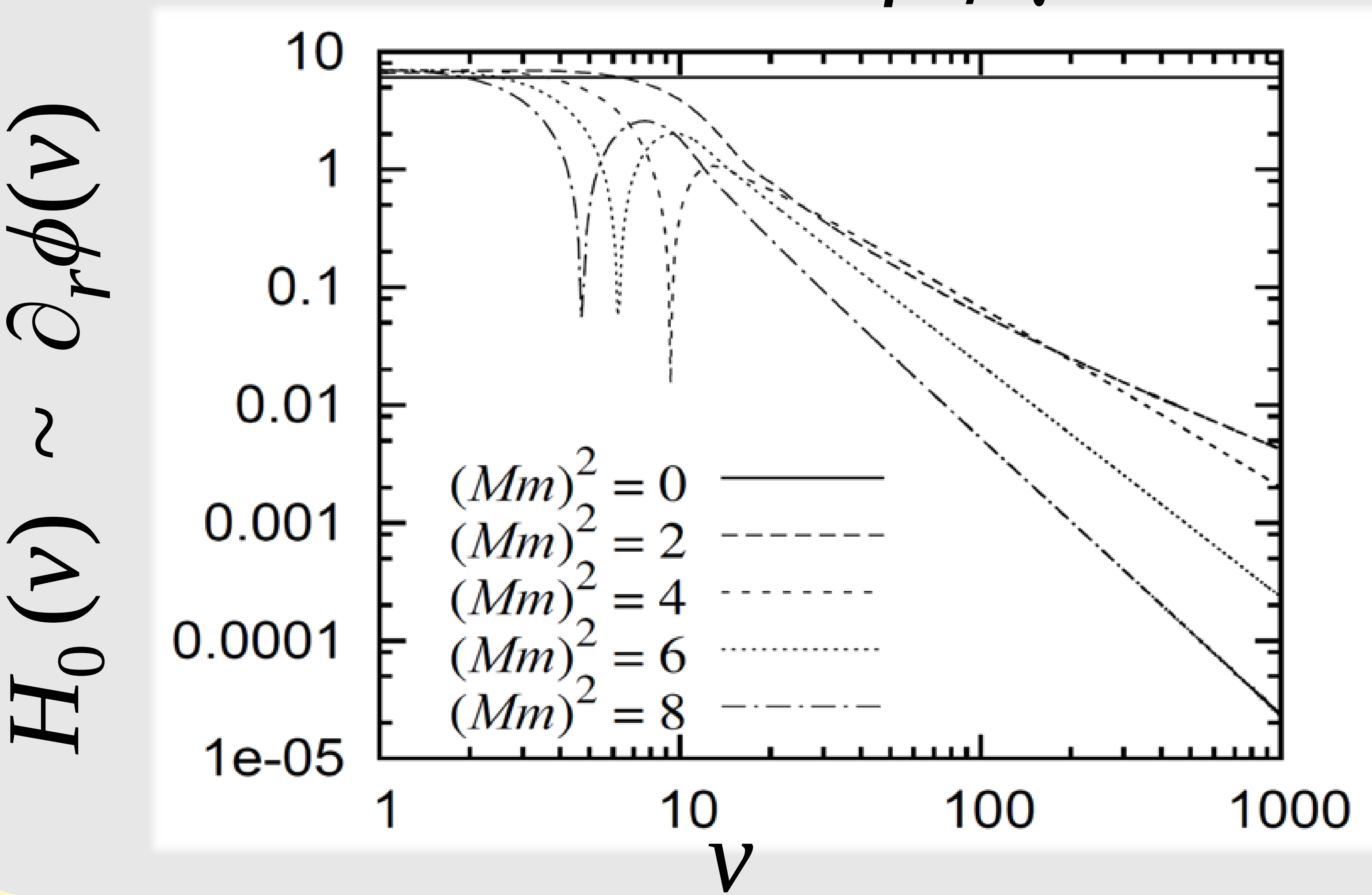
2012: Generalizations

– $m^2 = 2$



2012: Generalizations

– General $m^2 = n(n+1)$: $\partial_r{}^k \phi \; \mu \; v^{-(n+1)+k}$



2012: Generalizations

3. Instability for coupled EM & Grav. Perturbations

Einstein + Maxwell + ~~Scalar~~

– Moncrief's gauge invariant equations

[Moncrief 1974]

i. Hamiltonian formulation

$$H = \int d^4x \left(\pi^{ij} g_{ij,t} + A_i E^i_{,t} - N\, C - N_i C^i - A_0 E^i_{,i} \right)$$

ii. Expand $O(\varepsilon^2)$

iii. Take δ(constraint) as canonical variables

$$(\delta q_i,\, \delta\pi_i), \quad (\delta C_i,\, \delta\pi^C_i)$$

$$\{\delta q_i,\, \delta C_j\} = 0, \quad \{\delta\pi_i,\, \delta C_j\} = 0, \quad \{\delta C_j,\, \delta\pi^C_i\} = \delta_{ij}$$

2012: Generalizations

3. Instability for coupled EM & Grav. Perturbations

$$2\partial_v\partial_r\psi + \partial_r\left(r^{-2}\Delta\partial_r\psi\right) = r^{-2}W(r)\psi$$

Ex.) $\ell = 1$, odd parity on extreme RN

$$\psi = \left(1 - \frac{2M}{3r}\right)\pi_f + \frac{r}{6}p_1 \qquad W(r) = 2\left(1 + \frac{2M^2}{r^2}\right)$$

$$\left(\begin{aligned} \pi_f &= \frac{1}{2}\int_{S^2} d\Omega\left(\hat{e}^i\right)^* \delta A_i \\ p_1 &= \int_{S^2} d\Omega\left(\hat{e}^i\right)^* \left(\partial_v\delta g_{ri} - \partial_r\delta g_{vi} + \frac{2}{r}\delta g_{vi}\right) \end{aligned}\right)$$

2012: Generalizations

3. Instability for coupled EM & Grav. Perturbations

$$2\partial_v\partial_r\psi + \partial_r\left(r^{-2}\Delta\partial_r\psi\right) = r^{-2}W(r)\psi$$

Þ $$H_p[\psi] \equiv \partial_r^p\left(r^2 f\partial_r\psi\right)\Big|_{r=M} = \text{const.}$$

Þ $$\begin{cases} \partial_r^4\psi \propto H_2\, v & (\ell = 1) \\ \partial_r^{\ell+2\pm1}\psi_\pm \propto H_{\ell\pm1} v & (\ell \geq 2) \end{cases}$$

2013: Backreaction

- ≤ 2012:
  - Linear perturbations
  - Non-decay & Blow-up on extreme horizon

- Backreaction?
  - Ø Non-decay & Blow-up à Singularity etc?
  - Ø Suppression by nonlinearity à non-extreme RN?

- Clarify by solving the full Einstein equations

2013: Backreaction

- Gravity + Maxwell + Scalar:

$$ds^2 = -f(u,v)dudv + r(u,v)^2 d\Omega^2$$

$$\phi = \phi(u,v) \qquad\qquad A = \alpha(u,v)dv$$

- EoMs:

$$f_{,uv} = \frac{1}{2r^4 f}\left(2r^4 f_{,u} f_{,v} + r^2 f^3 + 4r^2 f^2 r_{,u} r_{,v} - 2Q^2 f^3 - r^4 f^2 \phi_{,u}\phi_{,v}\right)$$

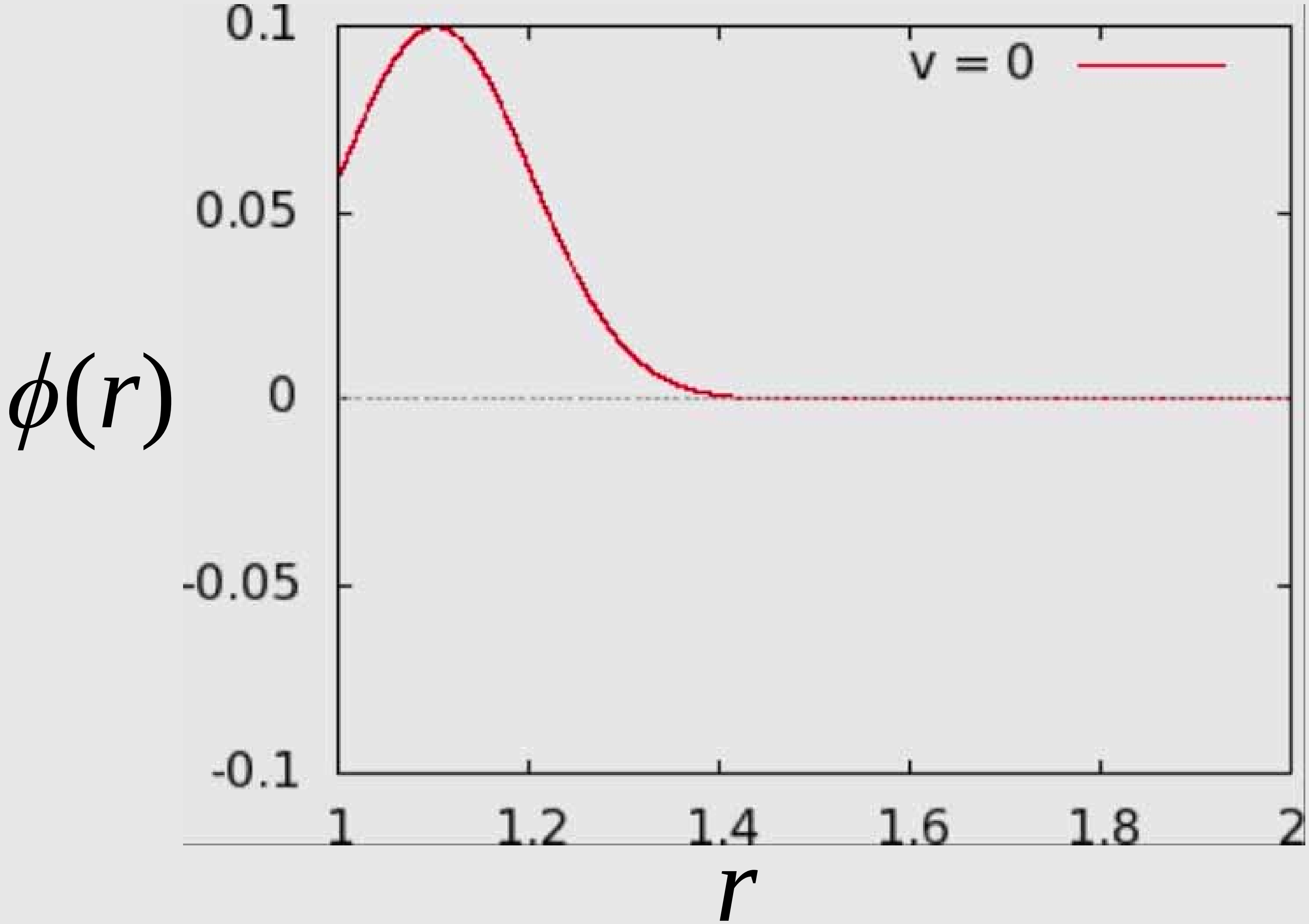
$$r_{,uv} = -\frac{1}{16r^3}\left(16r^2 r_{,u} r_{,v} + 4r^2 f - 4Q^2 f - m^2 r^4 f \phi^2\right)$$

$$\phi_{,uv} = -\frac{1}{4r}\left(4r_{,u}\phi_{,v} + 4r_{,v}\phi_{,u} + m^2 r f \phi\right) \qquad\qquad \alpha_{,u} = \frac{Qf}{r^2}$$

- Initial data = Extreme RN + Small wave packet

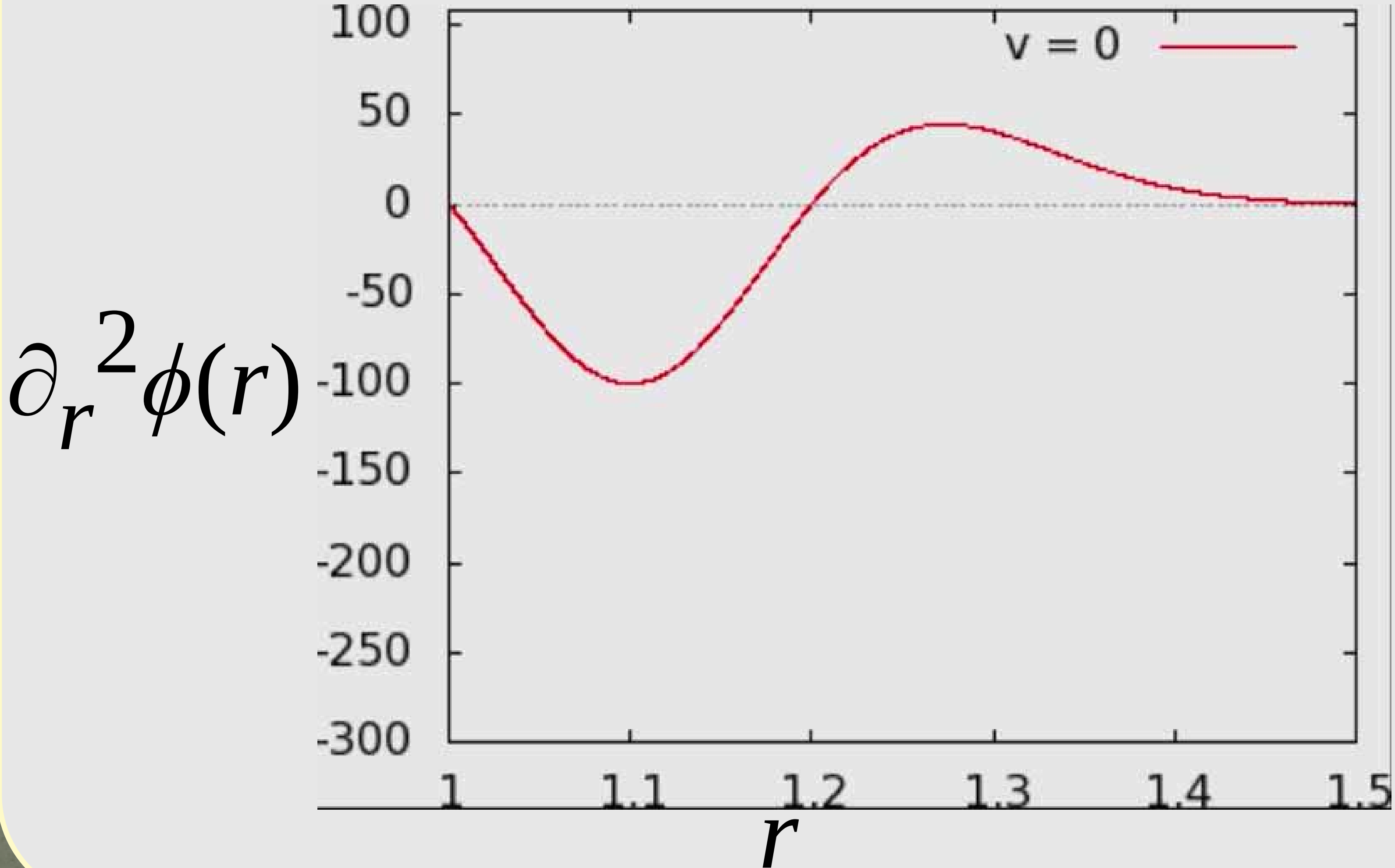
2013: Backreaction

– With backreaction



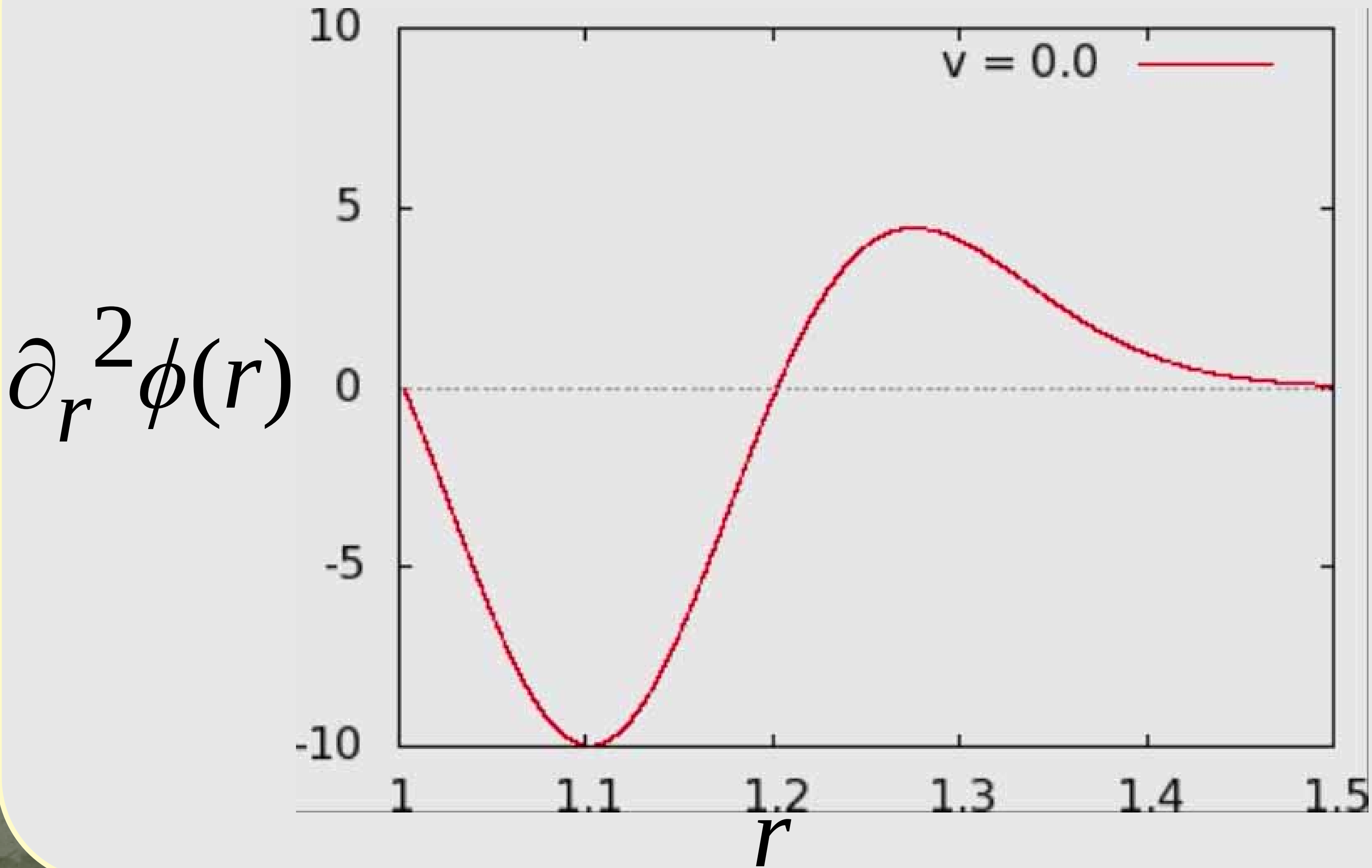
2013: Backreaction

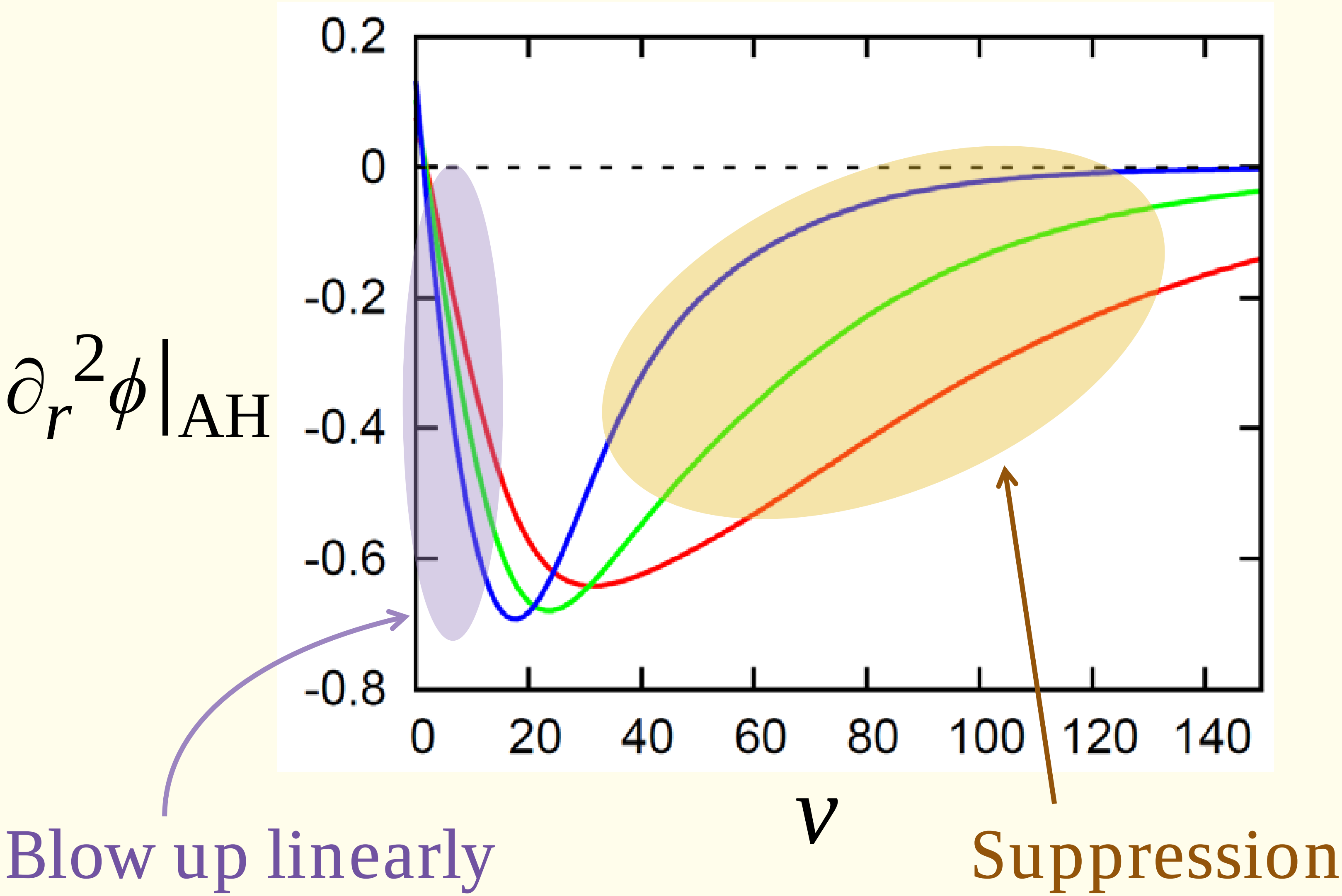
– ***No*** backreaction



2013: Backreaction

– With backreaction




0.2
0
-0.2
-0.4
-0.6
-0.8
0 20 40 60 80 100 120 140
$\partial_r{}^2\phi|_{AH}$
v
Blow up linearly
Suppression

Summary

1957 – 2010	BH Stability & Price's law
2011	Instability of extreme horizons
2012	Generalizations
2013	Backreaction

- Linear Instability on extreme horizons is universal
- Backreaction:

  Linear blow-up à Non-linear suppression à Non-extreme

- Any other final state?
- Any physical applications?

– Conformal isometry

[Bizon & Friedrich 2012]
[Lucietti, Murata, Reall, NT 2012]

– Extreme RN has discrete conformal isometry

$$\Phi: \ (t,r,\theta,\phi) \to \left(t, r' = M + \frac{M^2}{r-M}, \theta, \phi\right)$$

$$\Phi_*(g) \to \left(\frac{M}{r-M}\right)^{-2} g$$

$$\tilde{\psi}(u,r,\theta,\phi) = \frac{M}{r-M}\psi(v' = u, r', \theta, \phi)$$

– Newman–Penrose constants on I^+

$$\psi_0 \sim \frac{f_0(u)}{r} + \frac{C_0}{r^2} + \cdots$$

– NP constants « Aretakis constants by Φ

– AdS_2 analysis [Lucietti, Murata, Reall, NT 2012]

– Extremal à Near-horizon geometry = AdS_2

– Massless scalar on AdS2

$$\psi \to -\frac{2H_0}{v^2}$$

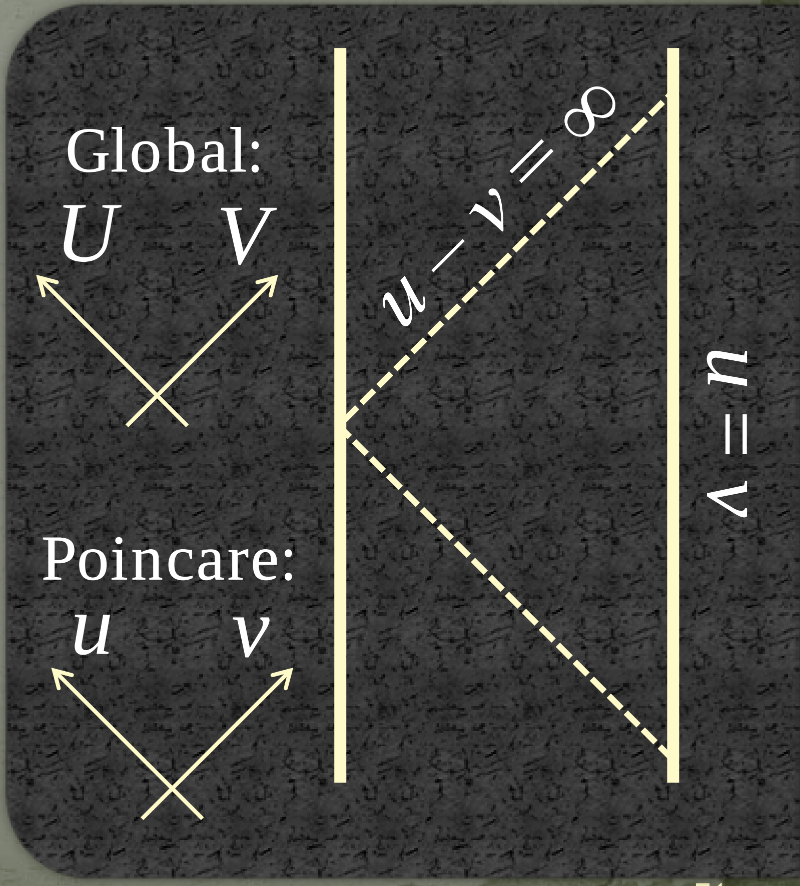
– Massive scalar

$$\psi \to (U - V)^{\Delta_m} F(V)$$

$$\Delta_m = \frac{1}{2} + \sqrt{m^2 + \frac{1}{4}}$$

è $\psi \sim v^{-(n+1)}$

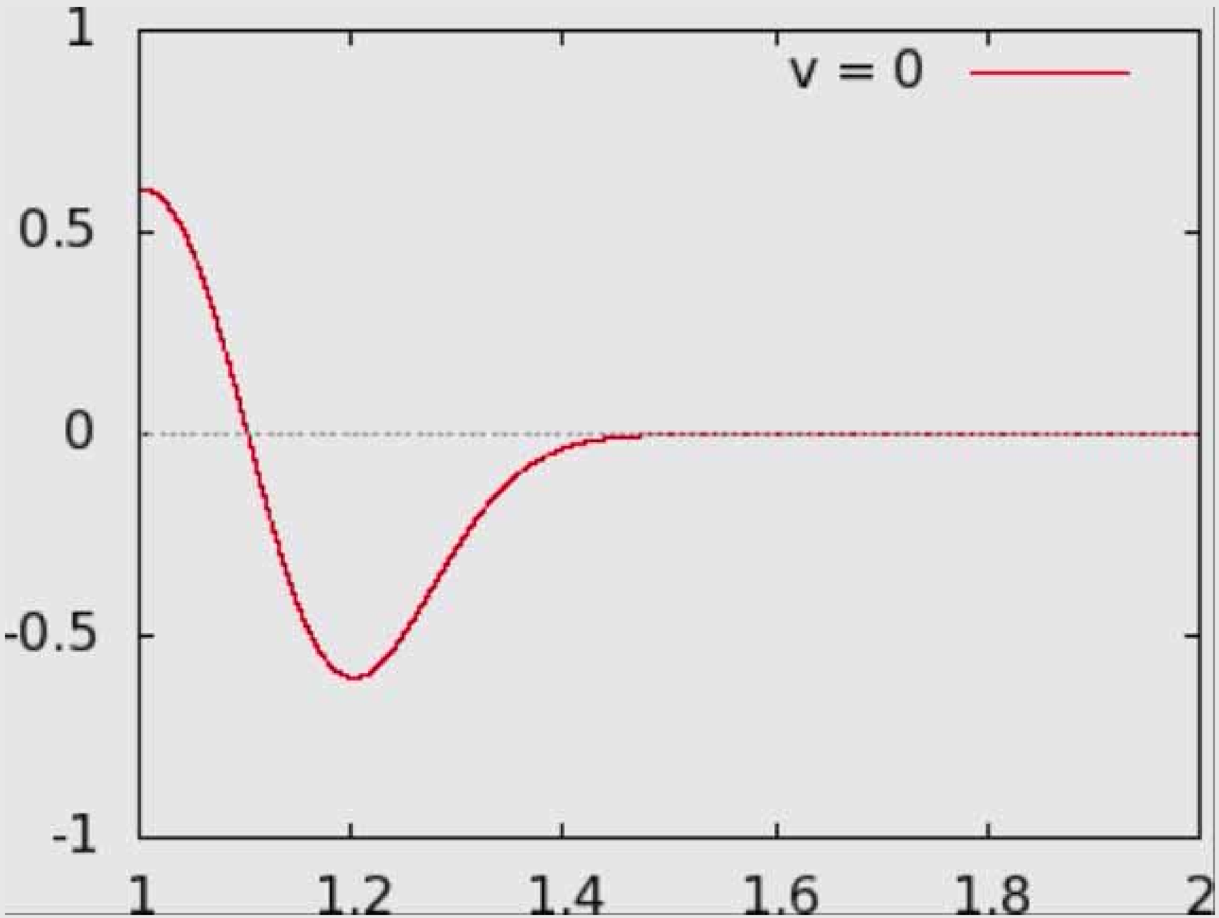
for $m^2 = n(n+1)$



2013: Backreaction

- Preliminary results:

$\partial_r \phi(r)$



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