Green Function Approach to Self-force Calculations

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[arXiv:1306.0884]

EMRIs

- A major goal of the space-based Gravitational Wave programme is to study Extreme Mass Ratio Inspirals.
- Many orbits.
- Expect generic (eccentric, inclined) orbits.
- Larger black hole generally spinning.
- Ultimate goal: ~10^4 accurate evolved generic orbits in Kerr with gravitational self-force.

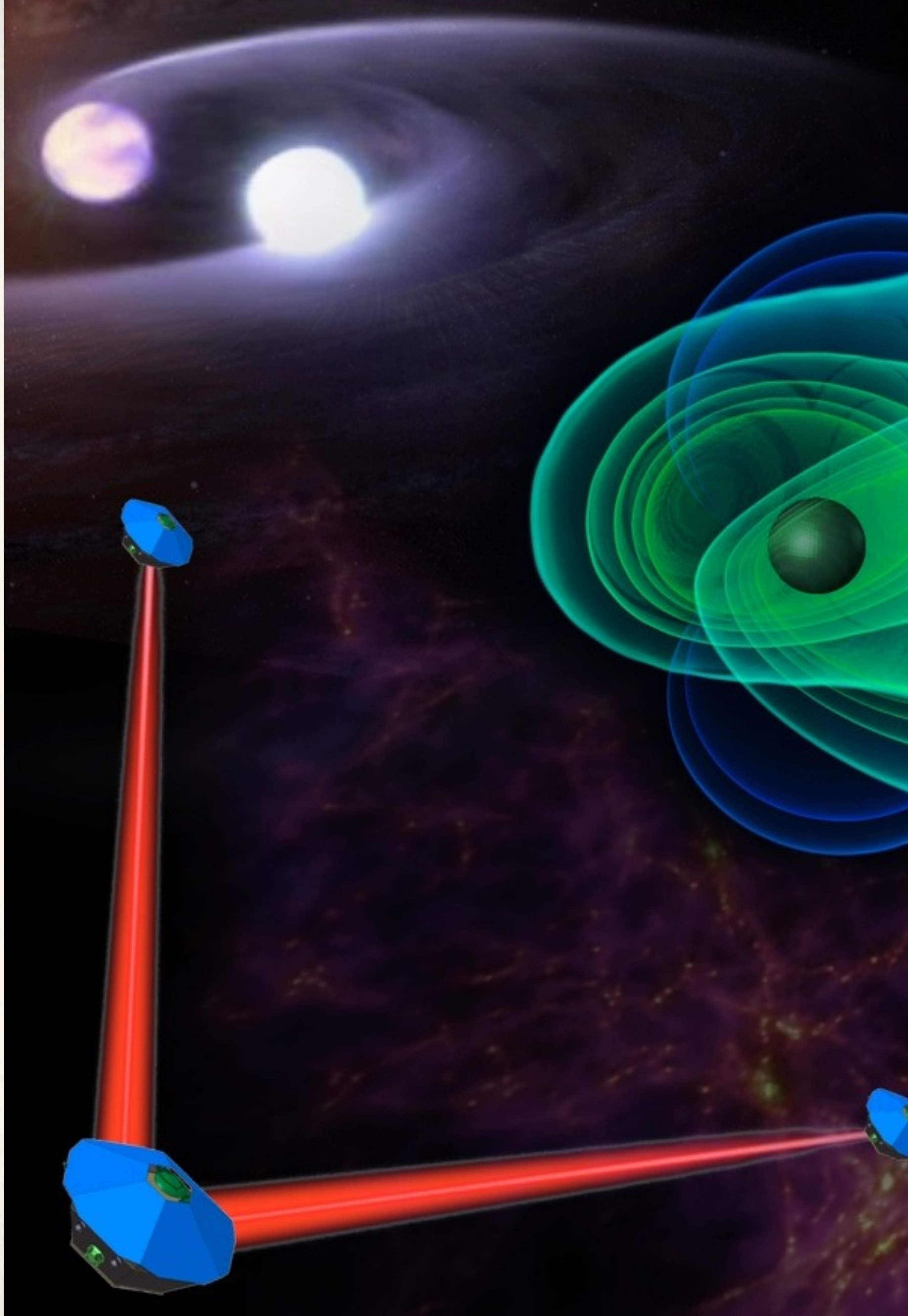


Image credit: eLISA/NGO Yellow book (ftp://ftp.rssd.esa.int/pub/ojennric/NGO_YB/NGO_YB.pdf)

Self-force

- Model the system using black hole perturbation theory => perturbative parameter is the mass/charge (*q*, *e*, *μ*≡*m*/*M*)
- Solve the coupled system of equations for the motion of a point particle and its retarded field.

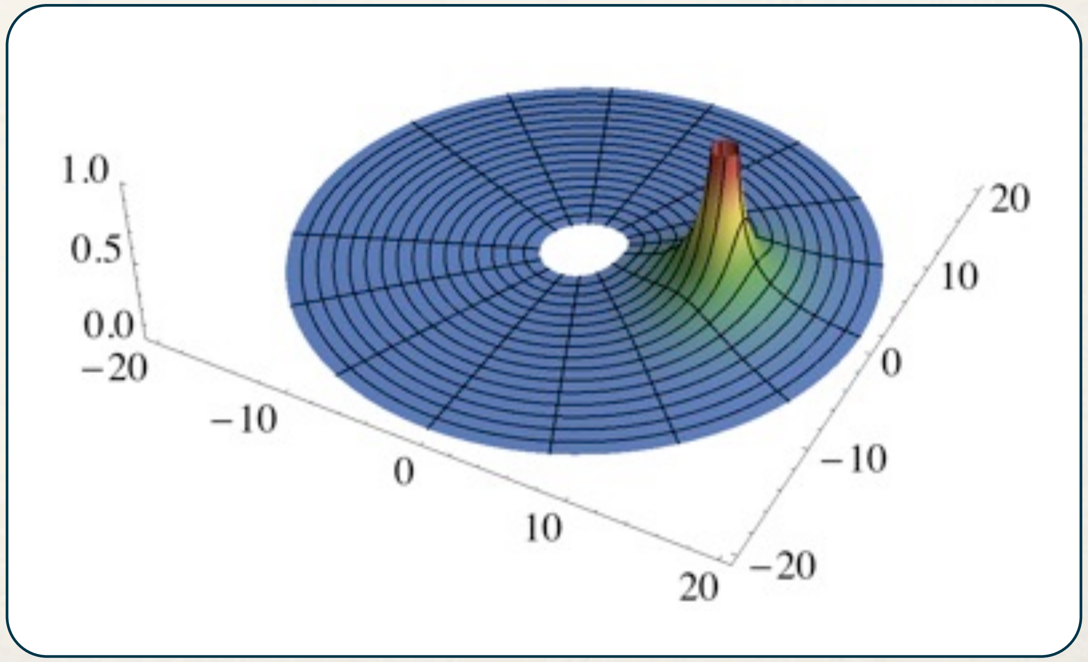
Scalar	Electromagnetic	Gravitational
$\Box\Phi^{\rm ret} = -4\pi q \int \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} d\tau$	$\Box A^{\rm ret}_a - R_a{}^b A^{\rm ret}_b = -4\pi e \int g_{aa'} u^{a'} \sqrt{-g}\delta_4(x, z(\tau)) d\tau$	$\Box\bar{h}^{\rm ret}_{ab} + 2C_a{}^c{}_b{}^d \bar{h}^{\rm ret}_{cd} = -16\pi\mu \int g_{a'(a} u^{a'} g_{b)b'} u^{b'} \sqrt{-g}\delta_4(x, z(\tau)) d\tau$
$\Phi^{\rm R} = \Phi^{\rm ret} - \Phi^{\rm S}$	$A^{\rm R}_a = A^{\rm ret}_a - A^{\rm S}_a$	$\bar{h}^{\rm R}_{ab} = \bar{h}^{\rm ret}_{ab} - \bar{h}^{\rm S}_{ab}$
$f_a = \nabla_a \Phi^{\rm R}$	$f^a = g^{ab} u^c A^{\rm R}{}_{[c,b]}$	$f^a = k^{abcd} \bar{h}^{\rm R}_{bc;d}$

$$a^\alpha = (g^{\alpha\beta} + u^\alpha u^\beta) f_\beta \qquad \frac{dm}{d\tau} = u^\beta f_\beta$$

Practical considerations

Several considerations arise when trying to turn this formal prescription into a practical calculation scheme:

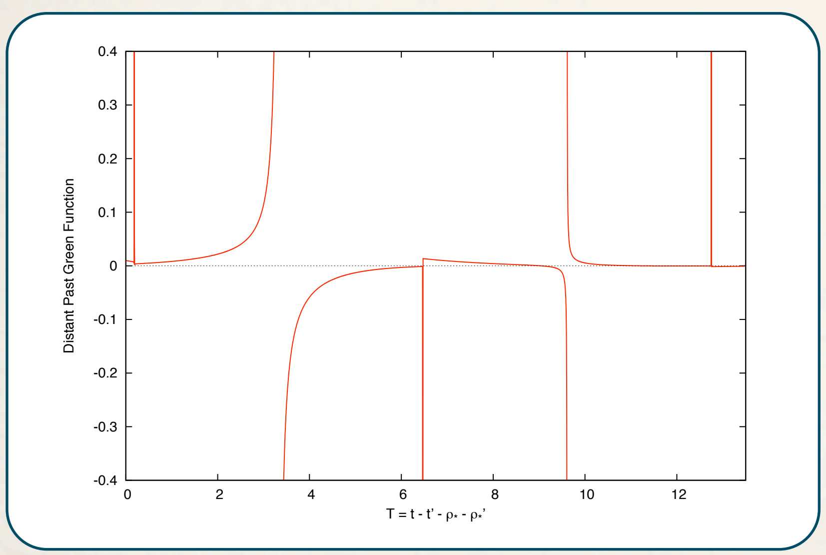
- System is coupled: retarded field depends on the entire past world-line and the world-line depends on field => delay differential equation.
- δ-function sources are difficult to handle numerically.
- Retarded field diverges like 1/r near the world-line.



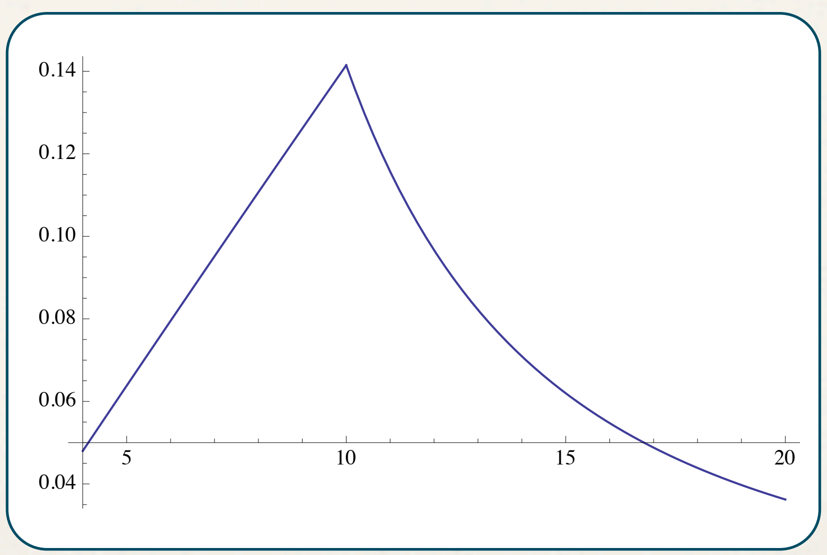
Approaches

- Several approaches have been developed for dealing with the numerical problems of point sources and singular fields.
- These broadly fall into three different categories

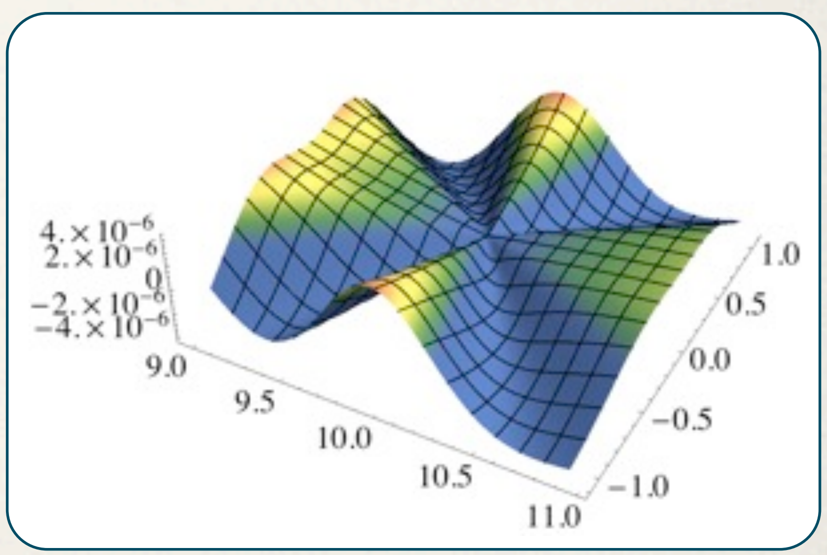
Green function



Mode-sum



Effective Source



Green function approach

Green function

- Solution of the wave equation with an impulsive source

$$\Box_x G(x, x') = \frac{4\pi}{\sqrt{-g}}\delta^4(x - x')$$

- For self-force calculations, we work with the retarded Green function
- Given the Green function, we can compute solutions of the sourced wave equation by integrating the Green function against the source

$$\Phi(x) = q \int_\gamma G_{\mathrm{ret}}(x, z(\tau'))d\tau'$$

- But, this diverges on the world-line, $x = z(\tau)$

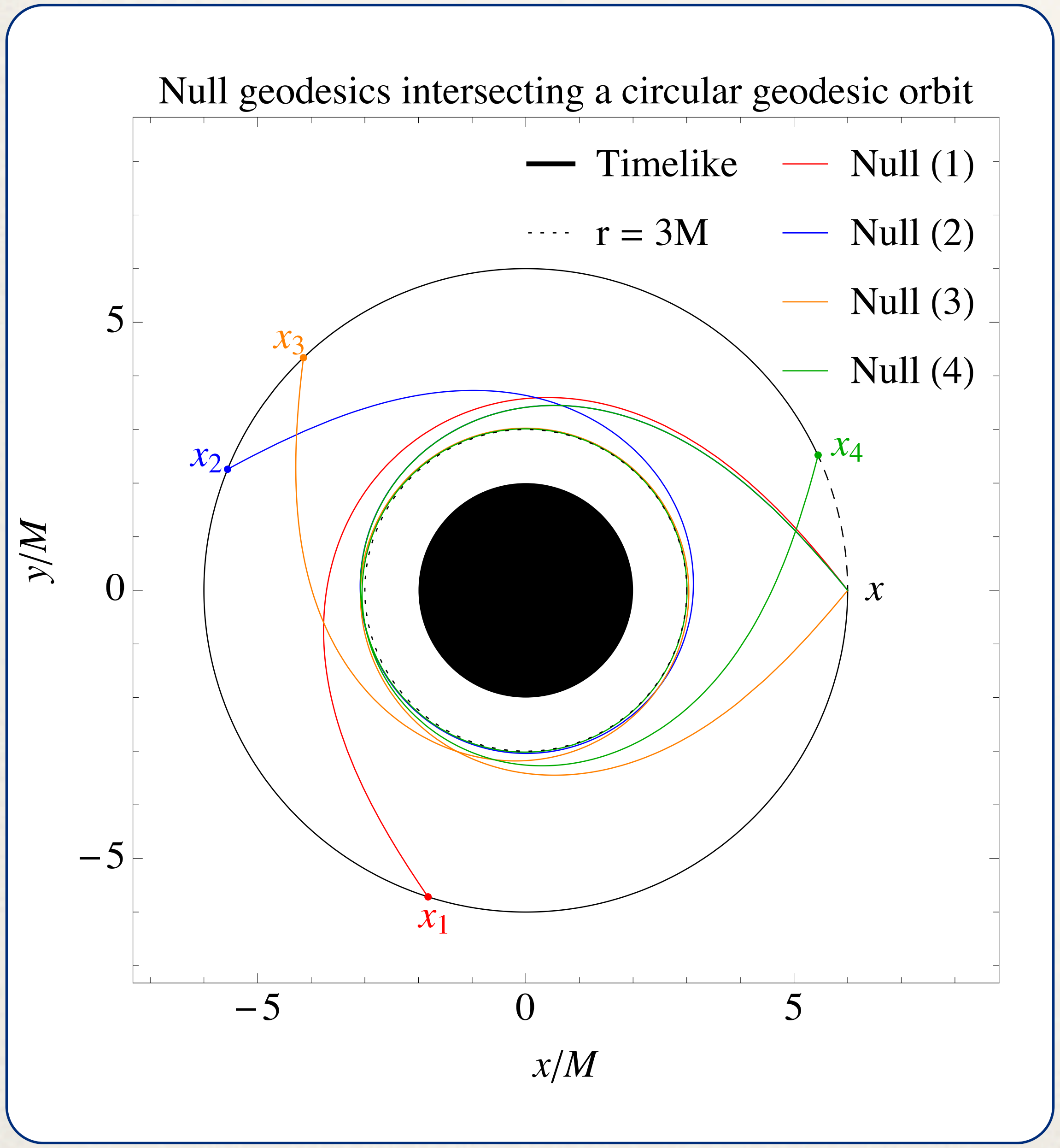
Green function regularization

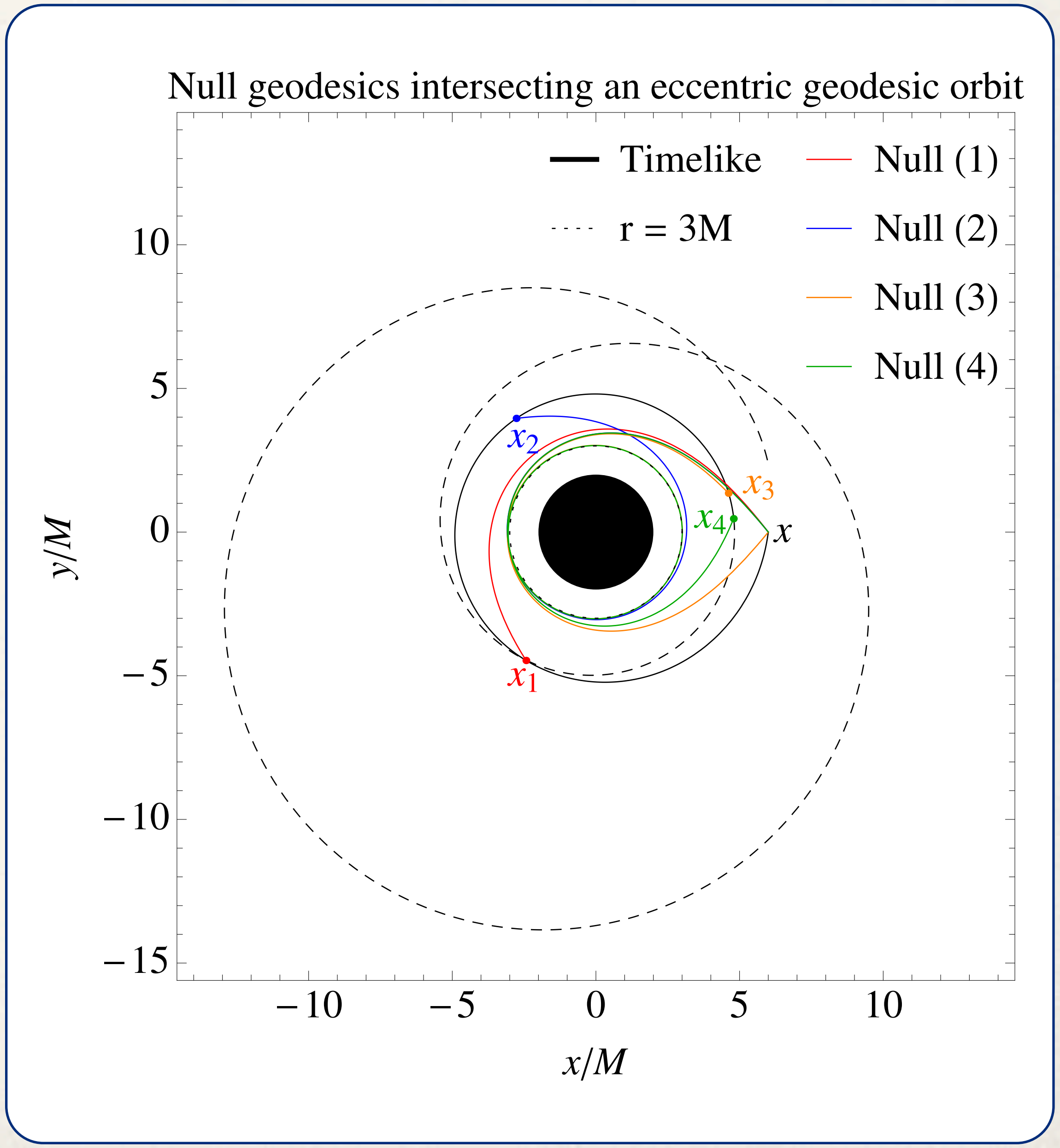
- Mino, Sasaki and Tanaka and Quinn and Wald derived an equation (MiSaTaQuWa) for the self-force in terms of a *tail* integral of the retarded Green function over the past world-line

$$f^a = (\text{local terms}) + \lim_{\epsilon \to 0} q^2 \int_{-\infty}^{\tau - \epsilon} \nabla^a G_{\text{ret}}(x, x') d\tau'$$

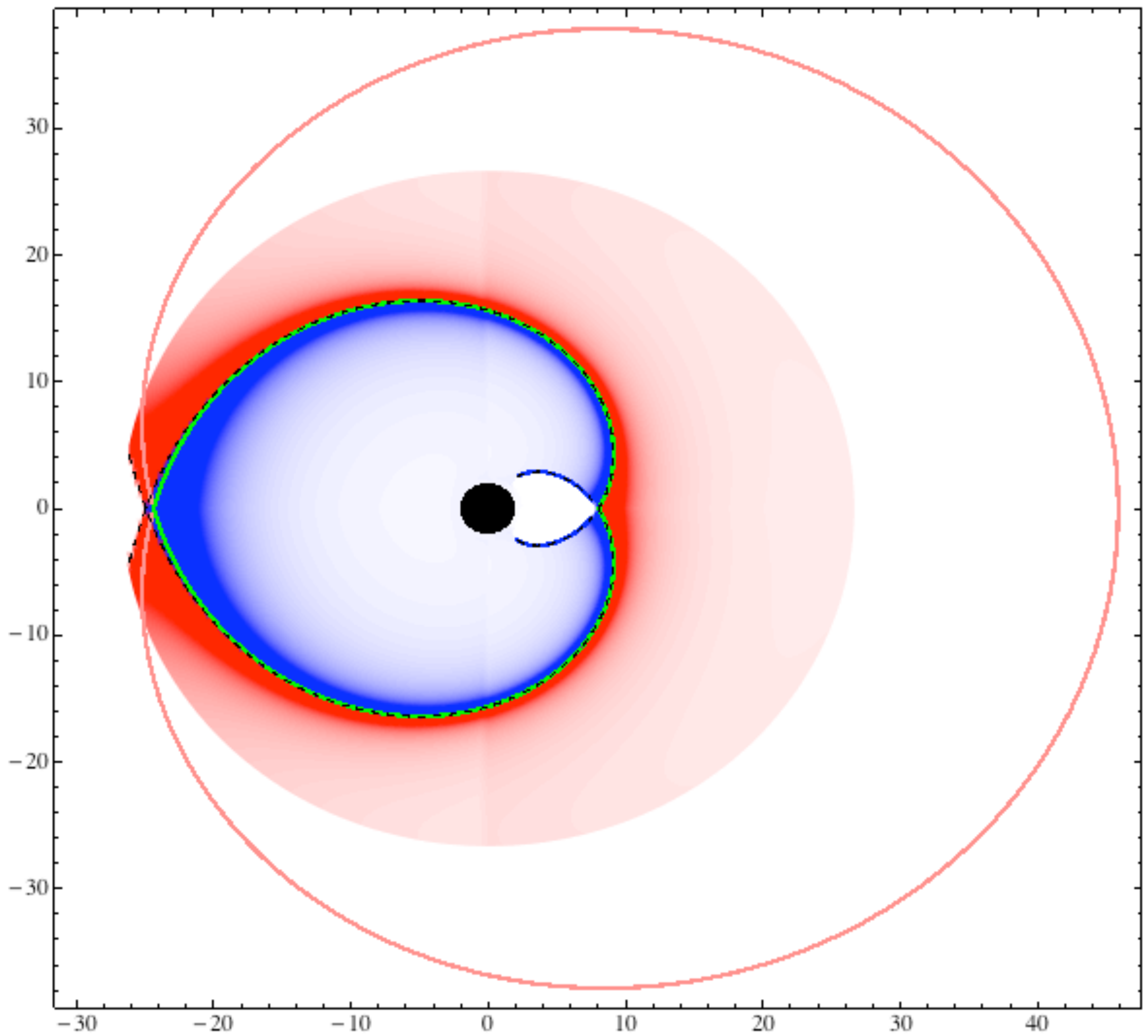
- The *tail* integral appears only in curved spacetime and contains information about the non-locality of the self-force.
- This can be understood geometrically in terms of null geodesics wrapping around the black hole and re-intersecting the world-line.

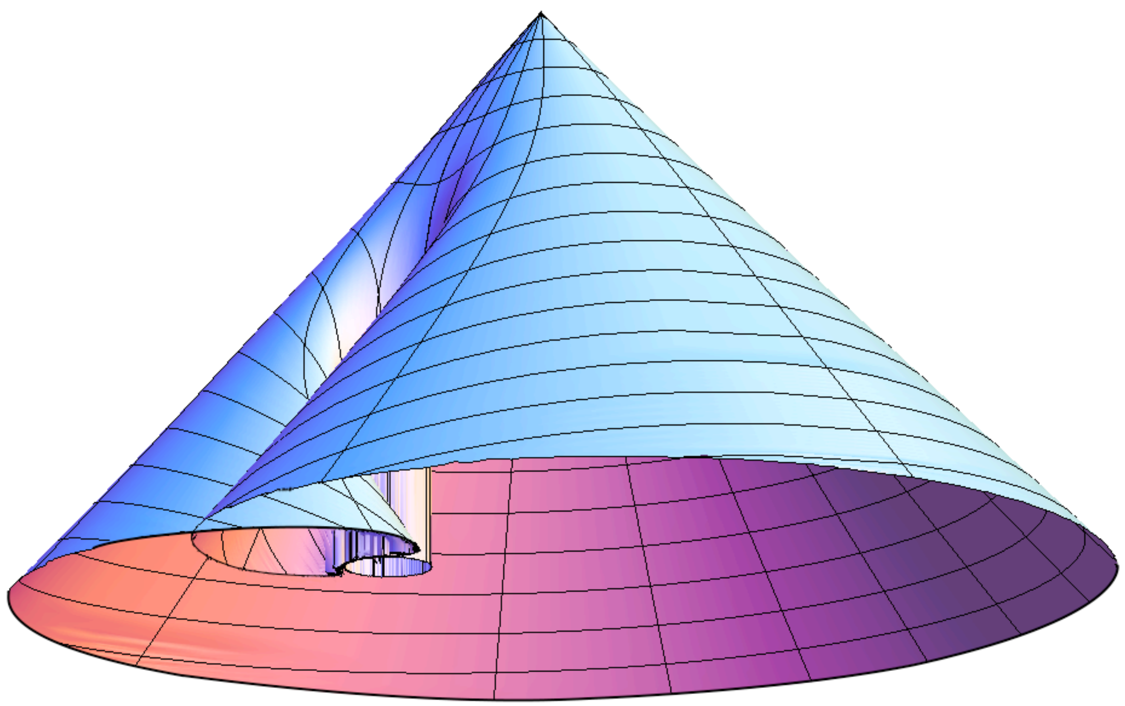
Tail contribution to the self-force

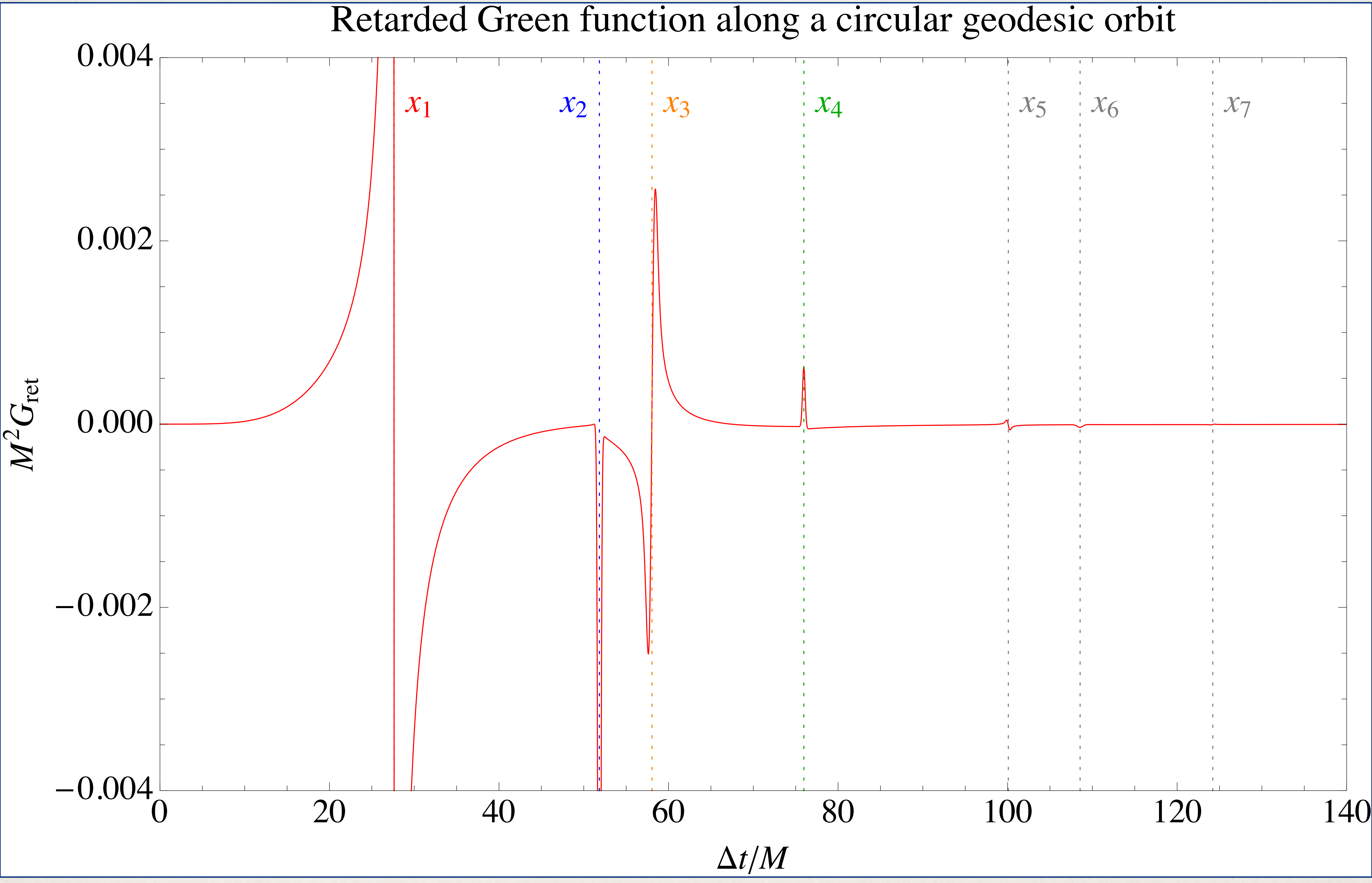


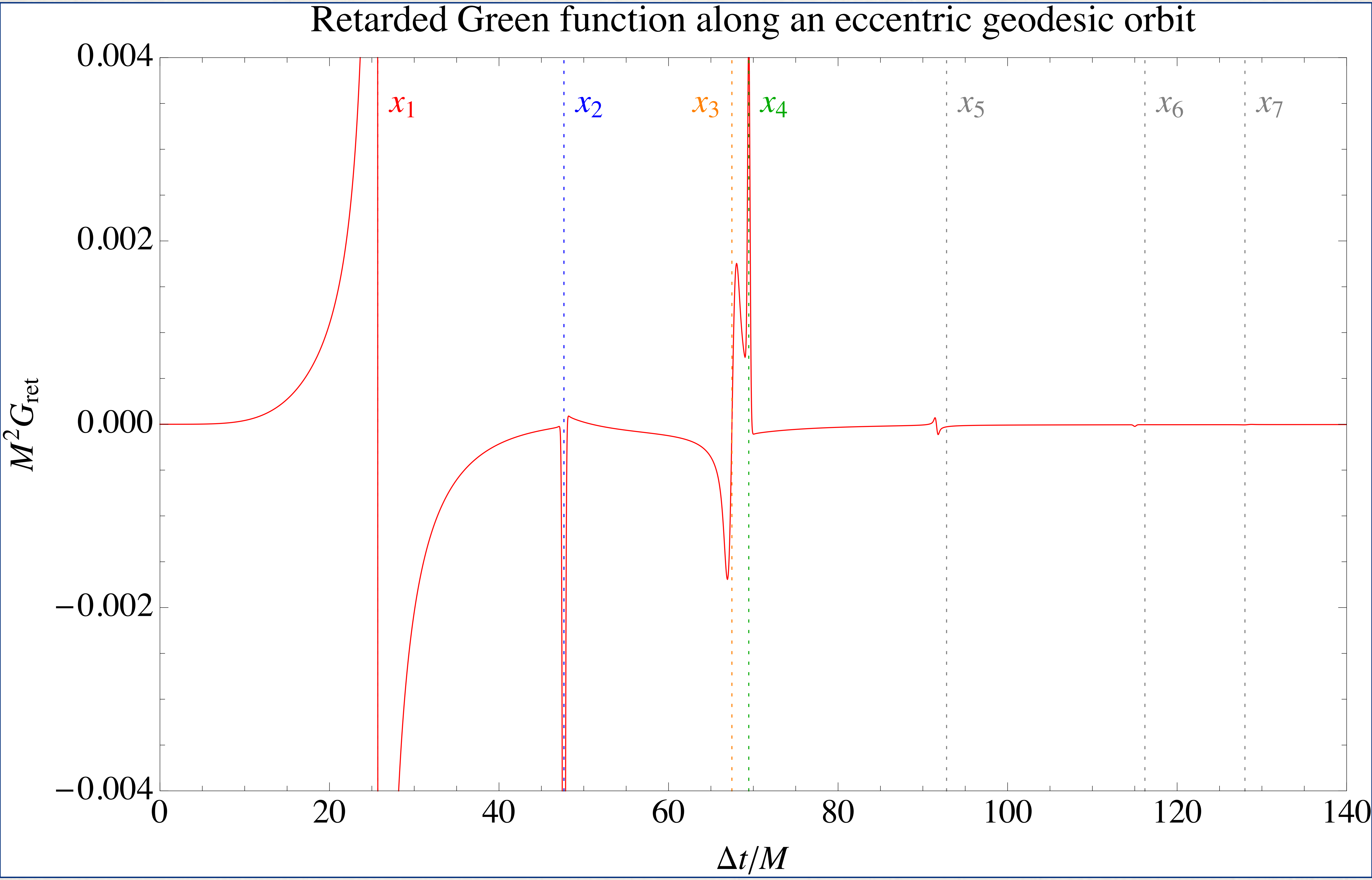


Tail contribution to the self-force






Retarded Green function along a circular geodesic orbit
0.004
0.002
0.000
−0.002
−0.004
$M^2 G_{\rm ret}$
x_1
x_2
x_3
x_4
x_5
x_6
x_7
0
20
40
60
80
100
120
140
$\Delta t/M$



Retarded Green function along an eccentric geodesic orbit
0.004
0.002
0.000
−0.002
−0.004
$M^2 G_{\rm ret}$
0
20
40
60
80
100
120
140
$\Delta t/M$
x_1
x_2
x_3
x_4
x_5
x_6
x_7

Green function regularization

- Mino, Sasaki and Tanaka and Quinn and Wald derived an equation (MiSaTaQuWa) for the self-force in terms of a *tail* integral of the retarded Green function over the past world-line

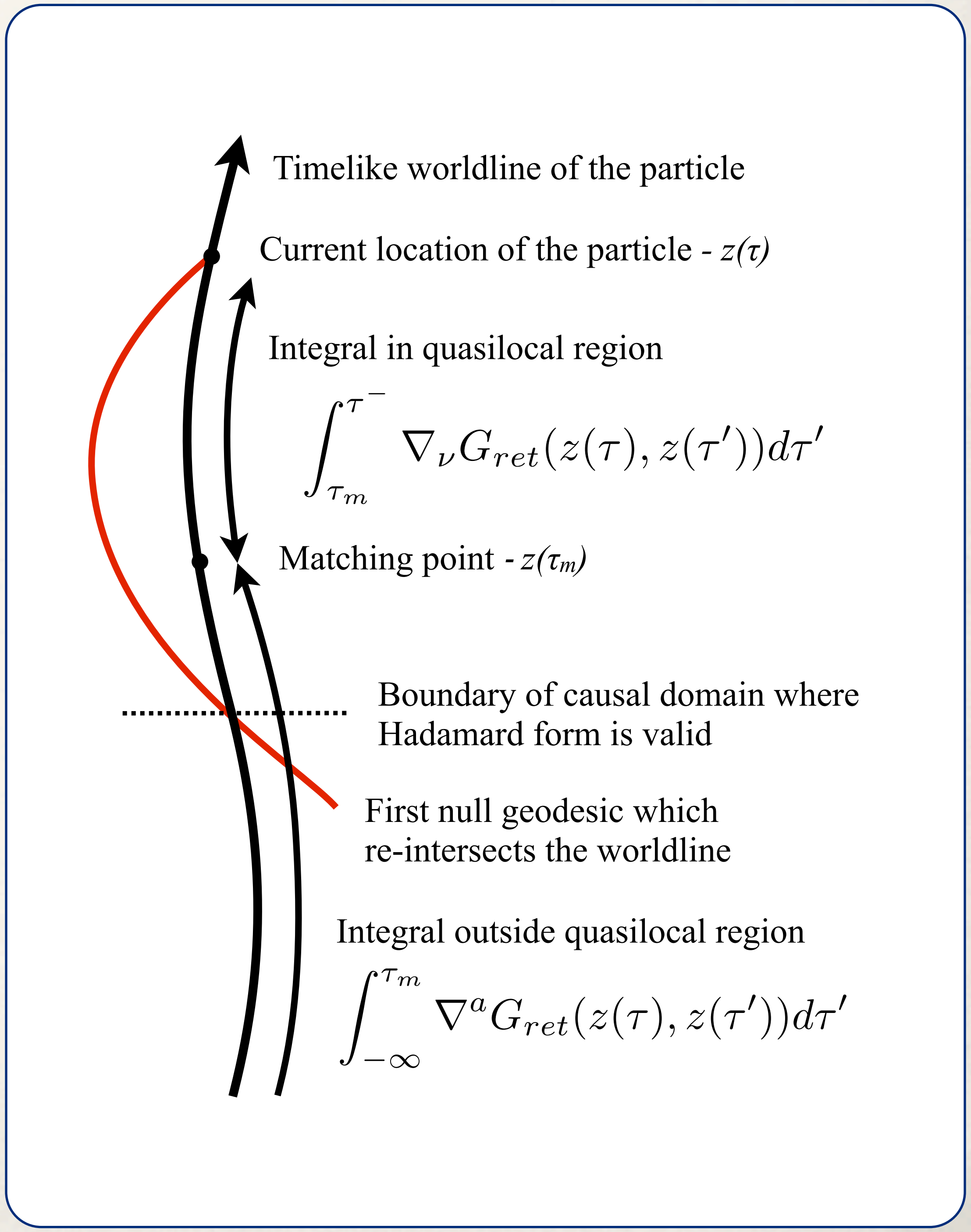
$$f^a = (\text{local terms}) + \lim_{\epsilon \to 0} q^2 \int_{-\infty}^{\tau - \epsilon} \nabla^a G_{\text{ret}}(x, x') d\tau'$$

- If we can compute the Green function along the world-line, then we're done: just integrate this to get the regularized self-force for any orbit.
- The difficulty is in developing a strategy for computing the Green function over a sufficiently large portion of the world-line.

Matched expansions calculation of the Green function

Matched expansions

- Compute the Green function using matched asymptotic expansions
  [Anderson and Wiseman, Class. Quantum Grav. 22, S783 (2005); M. Casals, S. R. Dolan, A. C. Ottewill, and B. Wardell, Phys. Rev. D 79, 124043 (2009)]
- Separately compute expansions of the Green function valid in the recent past (Taylor series) and in the distant past (quasi-normal mode + branch cut/ numerical time-domain evolution).
- Stitch together expansions in an overlapping matching region to give the full Green function.



Expansion in quasilocal region

Early times - quasilocal expansion

- For early times, we can use the Hadamard form

$$G(x,x') = \Theta_-(x,x')\left(U(x,x')\delta(\sigma) - V(x,x')\Theta(-\sigma)\right)$$

- Only need *V(x,x′)* since tail integral is inside the light-cone. Compute this as a series expansion (WKB) valid for *x* and *x′* close together; Padé re-summation to increase radius of convergence, accuracy.

$$V(x,x') = \sum_{i,j,k=0}^{\infty} v_{ijk}(r)\ (t-t')^{2i}(1-\cos\gamma)^j(r-r')^k$$

Expansion in distant past region

Late times - spectral expansion

- Spectral decomposition of the Green into spherical harmonic and Fourier modes

$$G^{\text{ret}}(x,x') = \sum_{\ell=0}^{\infty} \frac{1}{r\,r'}(2\ell+1)P_\ell(\cos\gamma)G_\ell^{\text{ret}}(r,r';\Delta t)$$

$$G_\ell^{ret}(r,r';\Delta t) \equiv \frac{1}{2\pi}\int_{-\infty+ic}^{\infty+ic} d\omega\ \tilde{G}_\ell(r,r';\omega)e^{-i\omega\Delta t}$$

$$\left[\frac{d^2}{dr_*^2}+\omega^2-V(r)\right]\tilde{G}_\ell(r,r';\omega) = -\delta(r_*-r'_*)$$

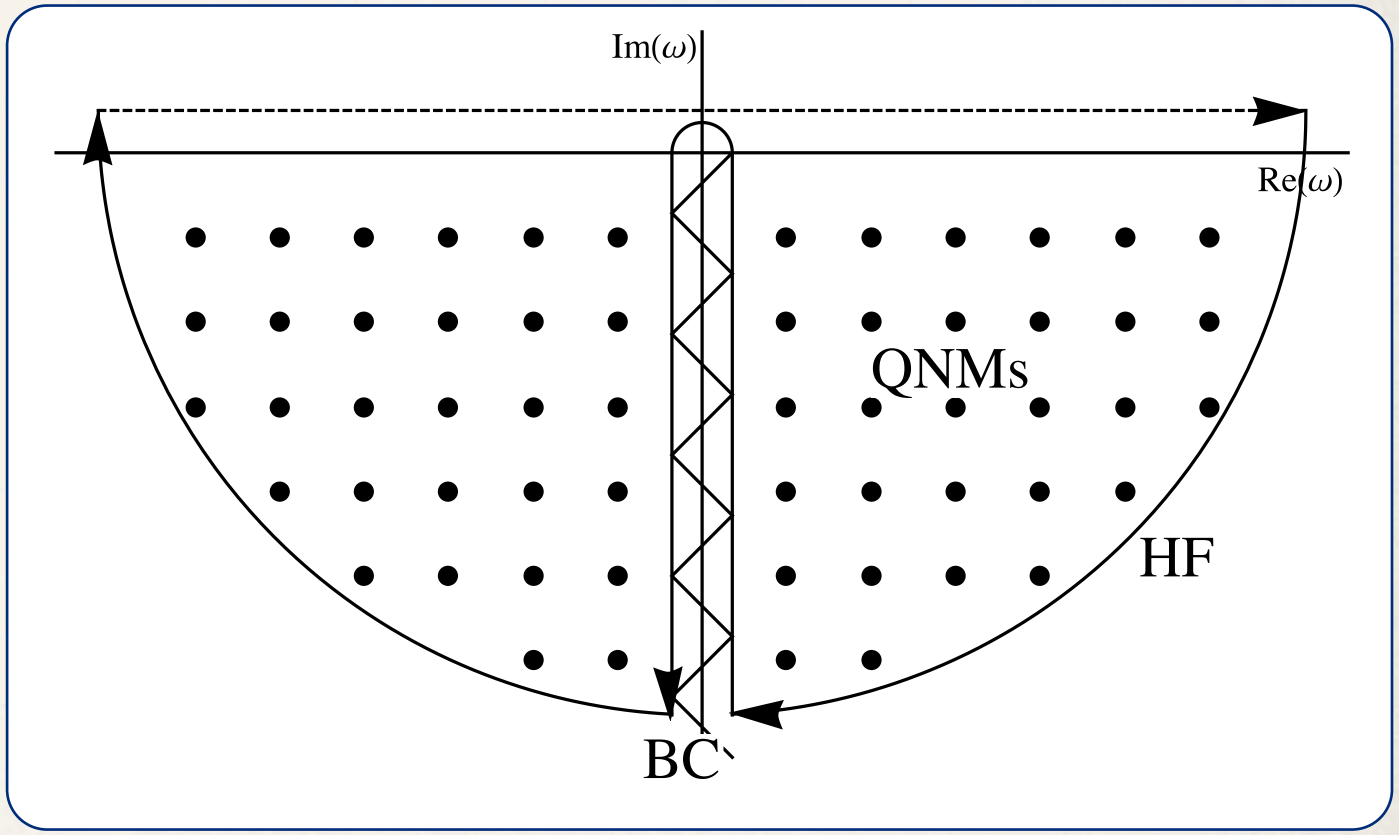
- Sum over *l* and integration over omega can be rendered convergent provided x and x' are far enough apart.

Late times - spectral expansion

- Integral over frequencies may be done by deforming the contour into the complex-frequency plane [Leaver (1988)].

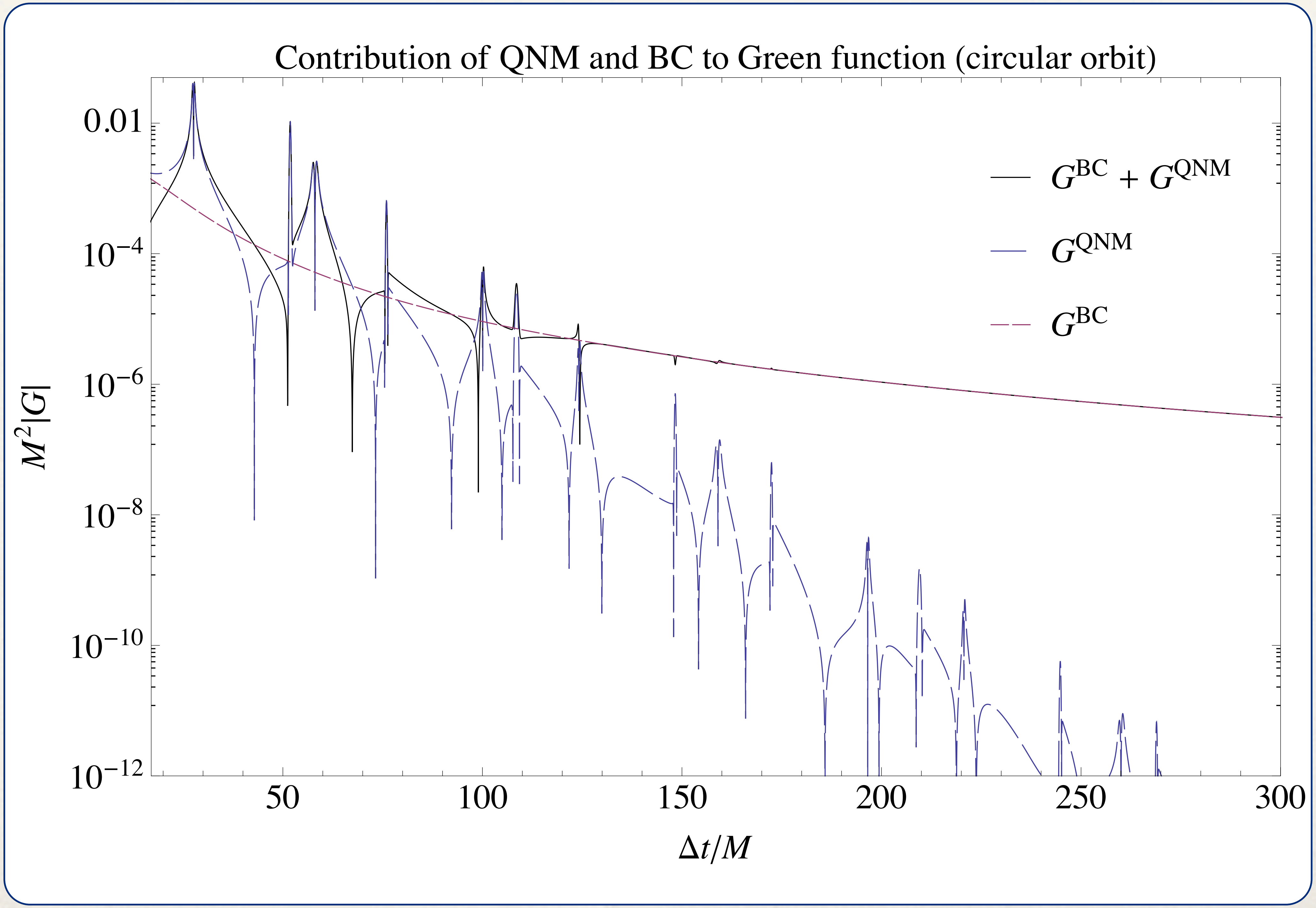
- Residue theorem of complex analysis dictates that one must account for singularities of the integrand.

- Simple poles (quasi-normal modes) and a branch cut down complex-frequency plane.

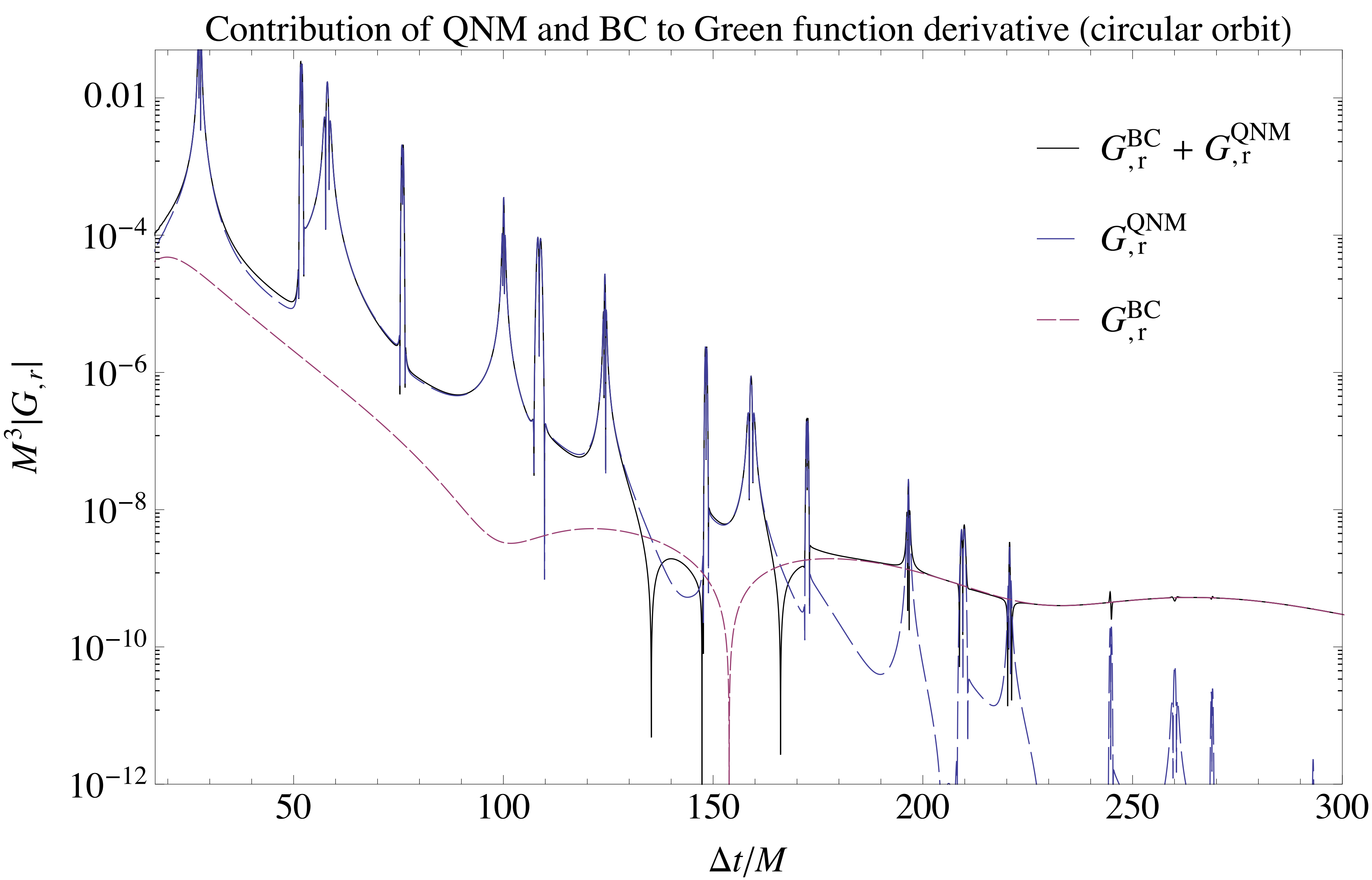


- High-frequency arc may be ignored; only contributes at “early” times.

Distant past Green function



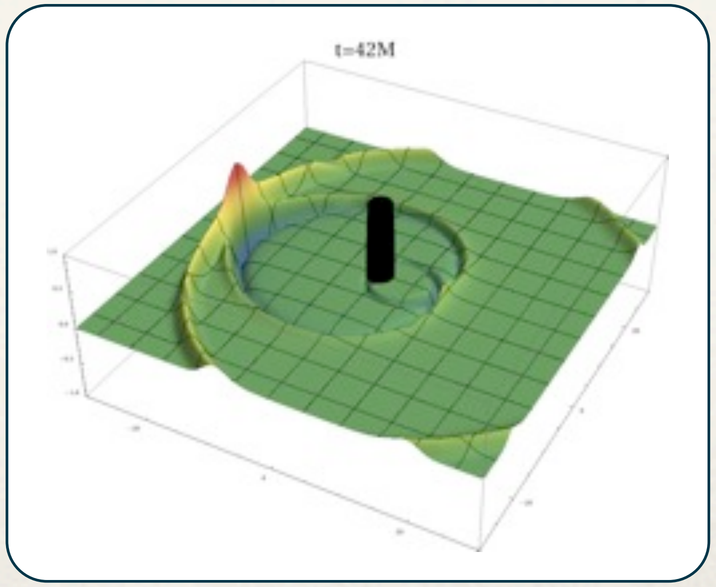
Distant past Green function



Numerical calculation in distant past region

Numerical time-domain evolution

- Alternative option is to numerically compute the retarded Green function using time-domain evolution.
- Numerically evolve a wave equation for the Green function.
- Still need to make use of quasi-local expansion for the recent past, but late time calculation is much easier than with quasi-normal modes + branch cut.



Numerical time-domain evolution

- Given initial data on a spatial hyper-surface Σ and the full Green function, can determine the solution at an arbitrary point x' in the future of Σ (Kirchhoff theorem)

$$\Phi(x') = -\frac{1}{4\pi}\int_\Sigma [G(x,x')\nabla^\alpha\Phi(x) - \Phi(x)\nabla^\alpha G(x,x')]d\Sigma_\alpha$$

- Basic idea: choose as initial data

$$\Phi(x) = 0 \qquad \partial_t\Phi(x) = \frac{4\pi}{(2\pi\sigma^2)^{3/2}}e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{2\sigma^2}}$$

then in the limit $\sigma \to 0$

$$\begin{aligned}\Phi(x') &= \int_\Sigma G(x,x')\delta_3(x-x_0)r^2\sin\theta dr d\theta d\phi \\ &= G(x_0,x').\end{aligned}$$

Numerical time-domain evolution

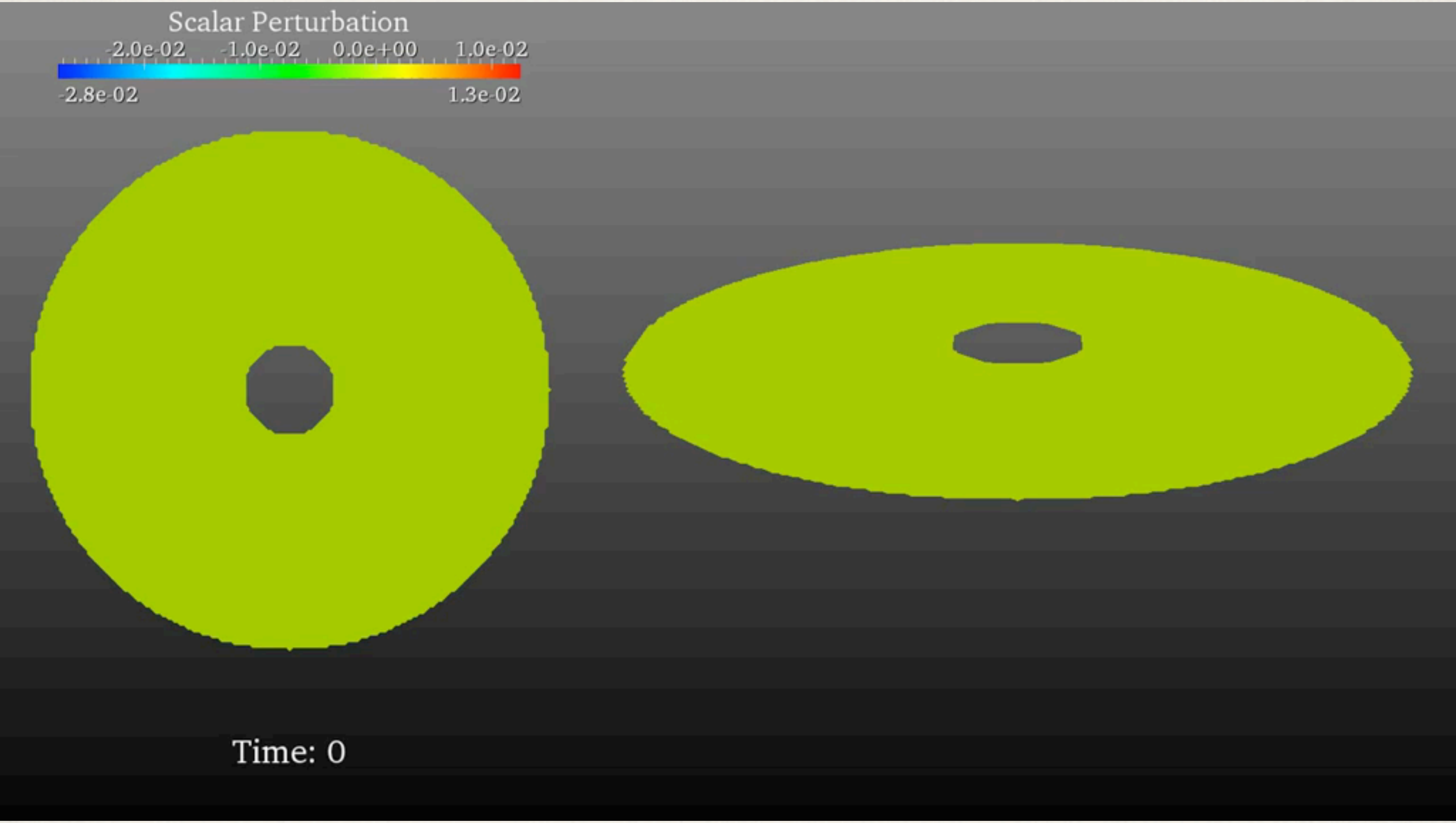
- So, we evolve the homogeneous wave equation with initial data

$$\Phi(x) = 0 \qquad \partial_t \Phi(x) = \frac{4\pi}{(2\pi\sigma^2)^{3/2}} e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{2\sigma^2}}$$

  for a sequence of values of σ then extrapolate to σ → 0 to get the Green function. Equivalent to smoothly cutting off the divergent sum over spherical-harmonic *l* modes, rendering it convergent.

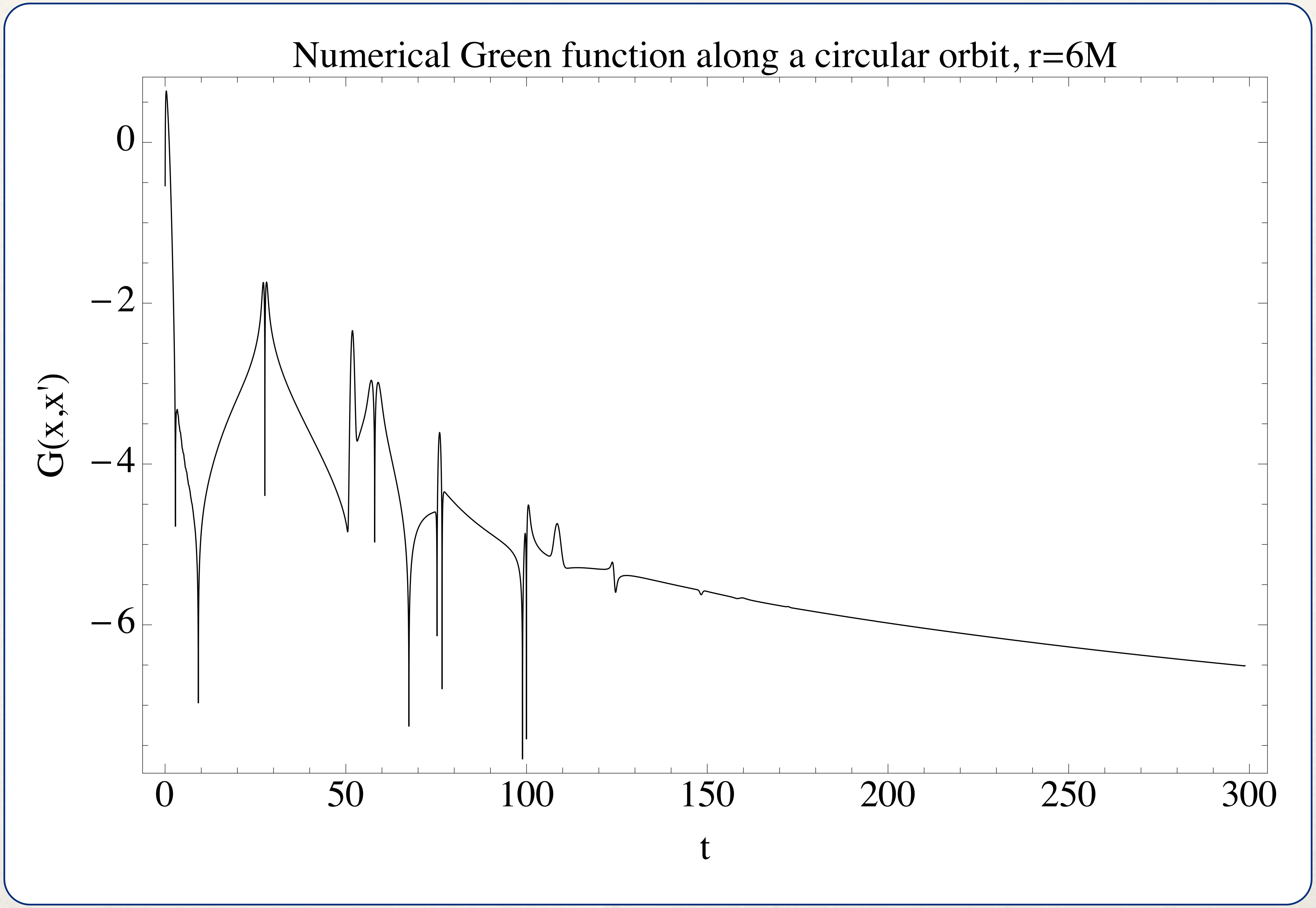
- Somewhat surprisingly, this works very well for computing the self-force, even for quite large σ/*M* ~ 0.1 - 1.

- Narrower Gaussian improves resolution of spikes at null-geodesic crossings. Between crossings, even a large σ is sufficient.



$G(x_0, x')$ for $x_0 = \{0, 6M, 0, \pi/2\}$ in Schwarzschild spacetime

Distant past Green function



Matching early- and late-time expansions

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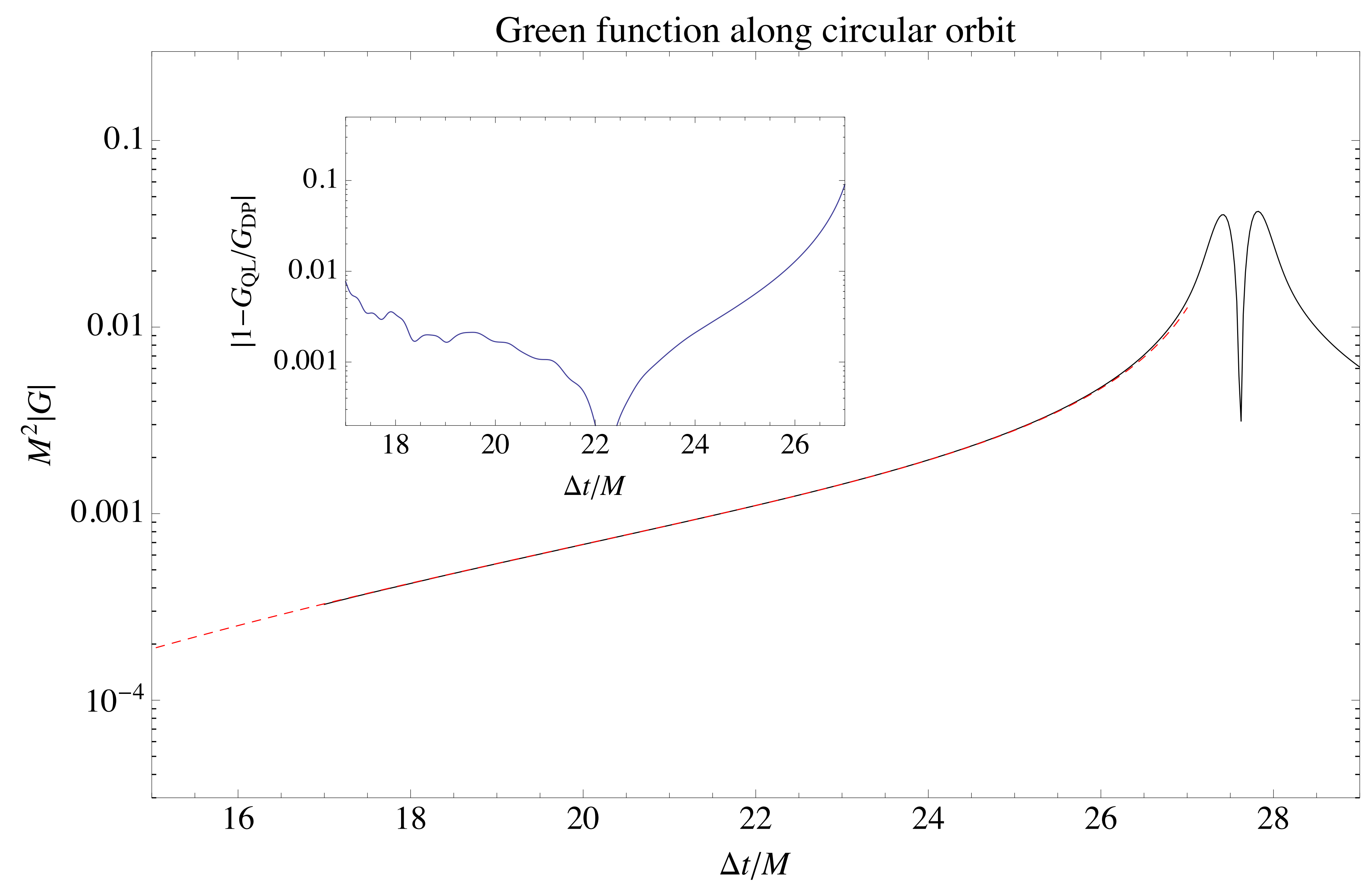
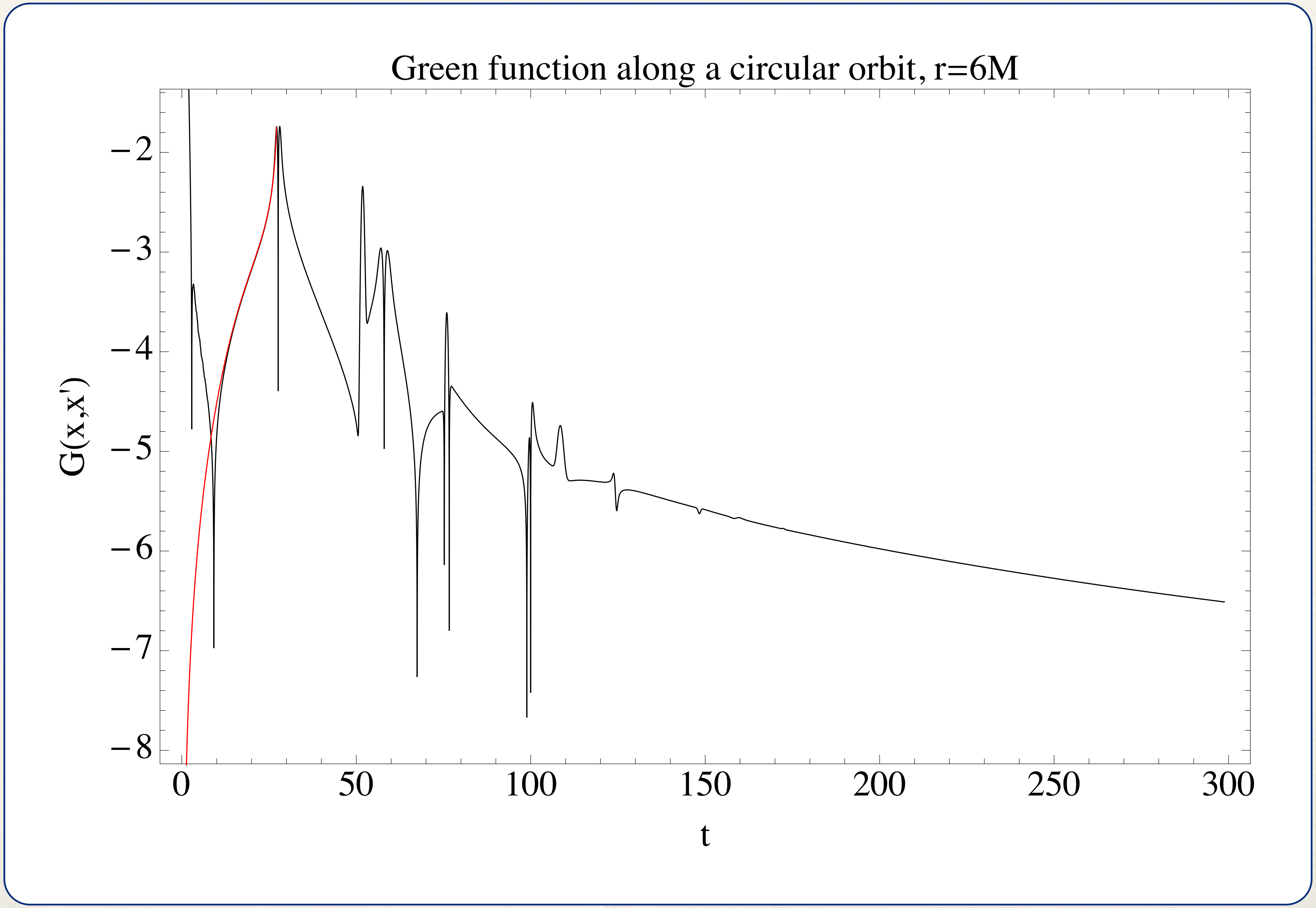
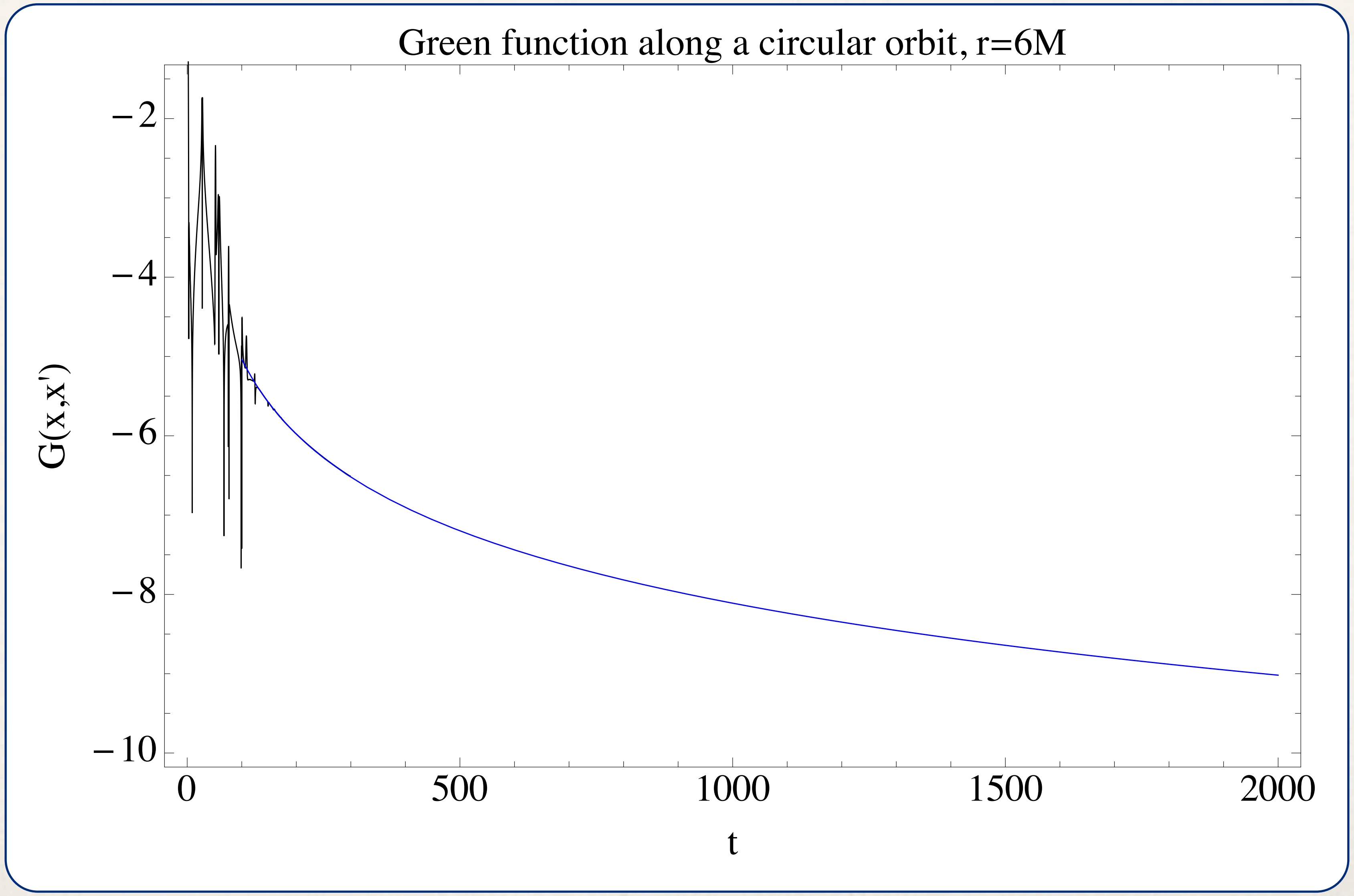


Early times - quasilocal expansion

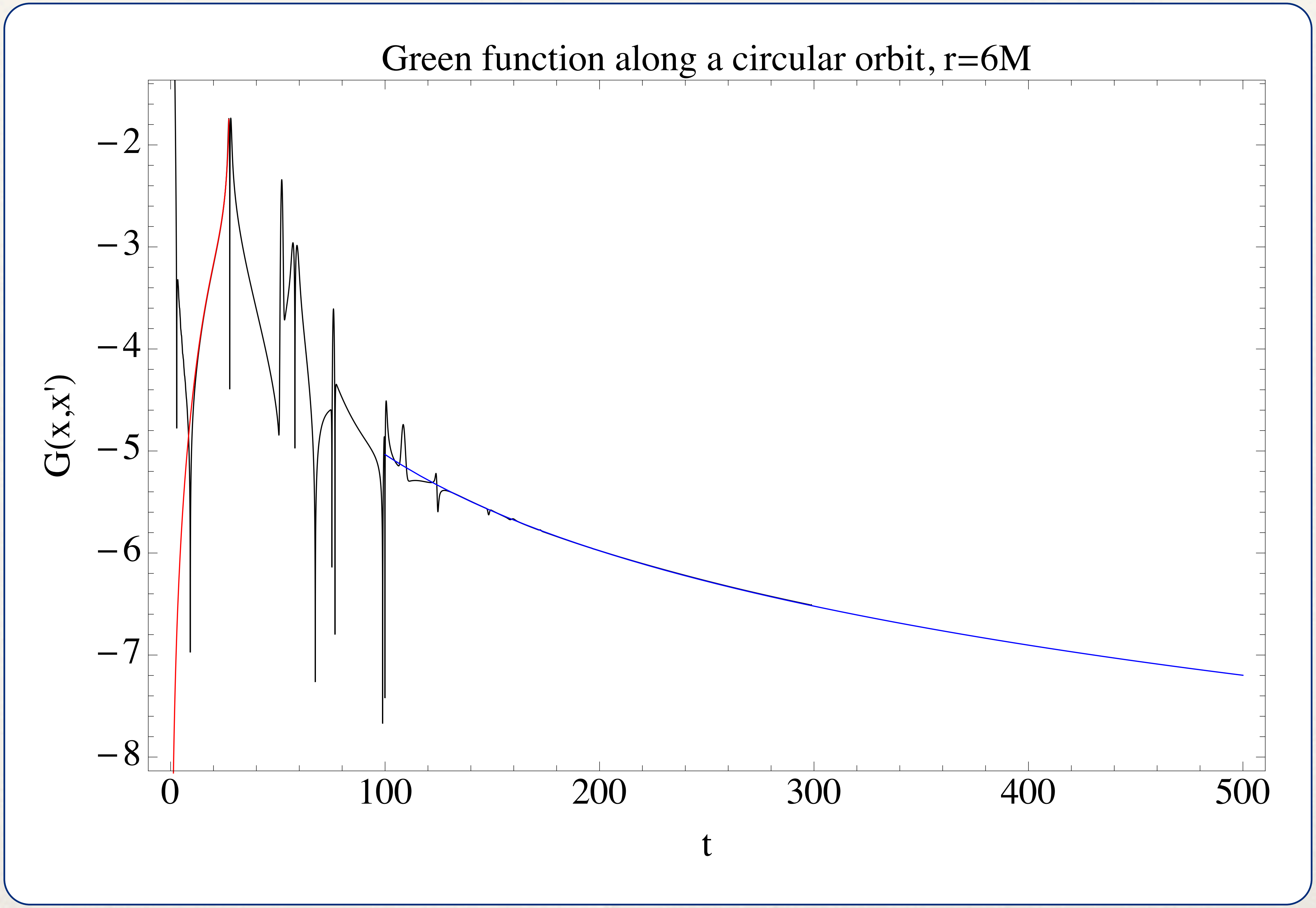


Late times - branch cut tail

- At very late times, Green function dominated by branch cut because quasi-normal modes decay much faster. Use analytic expressions.



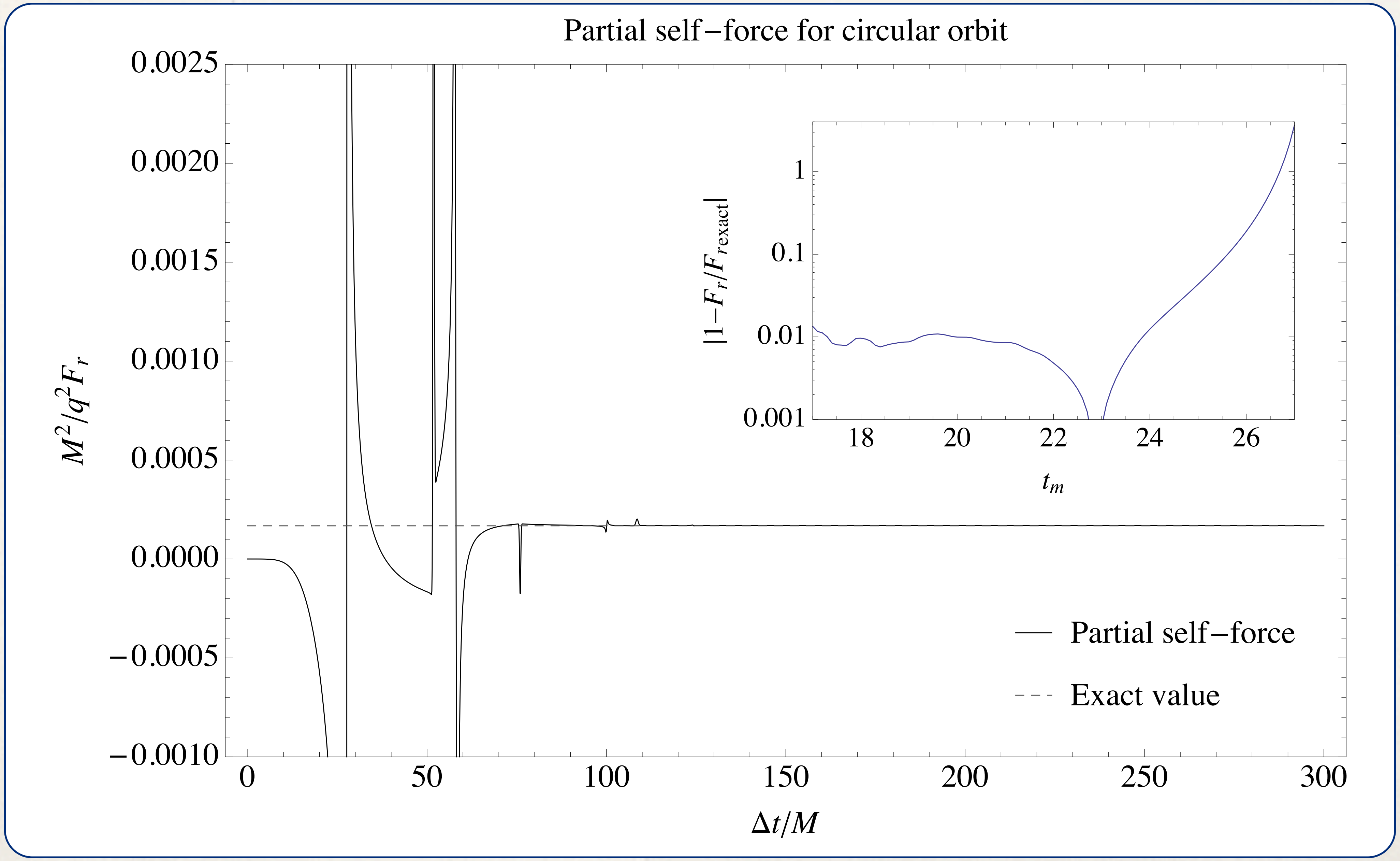
Matched Green function

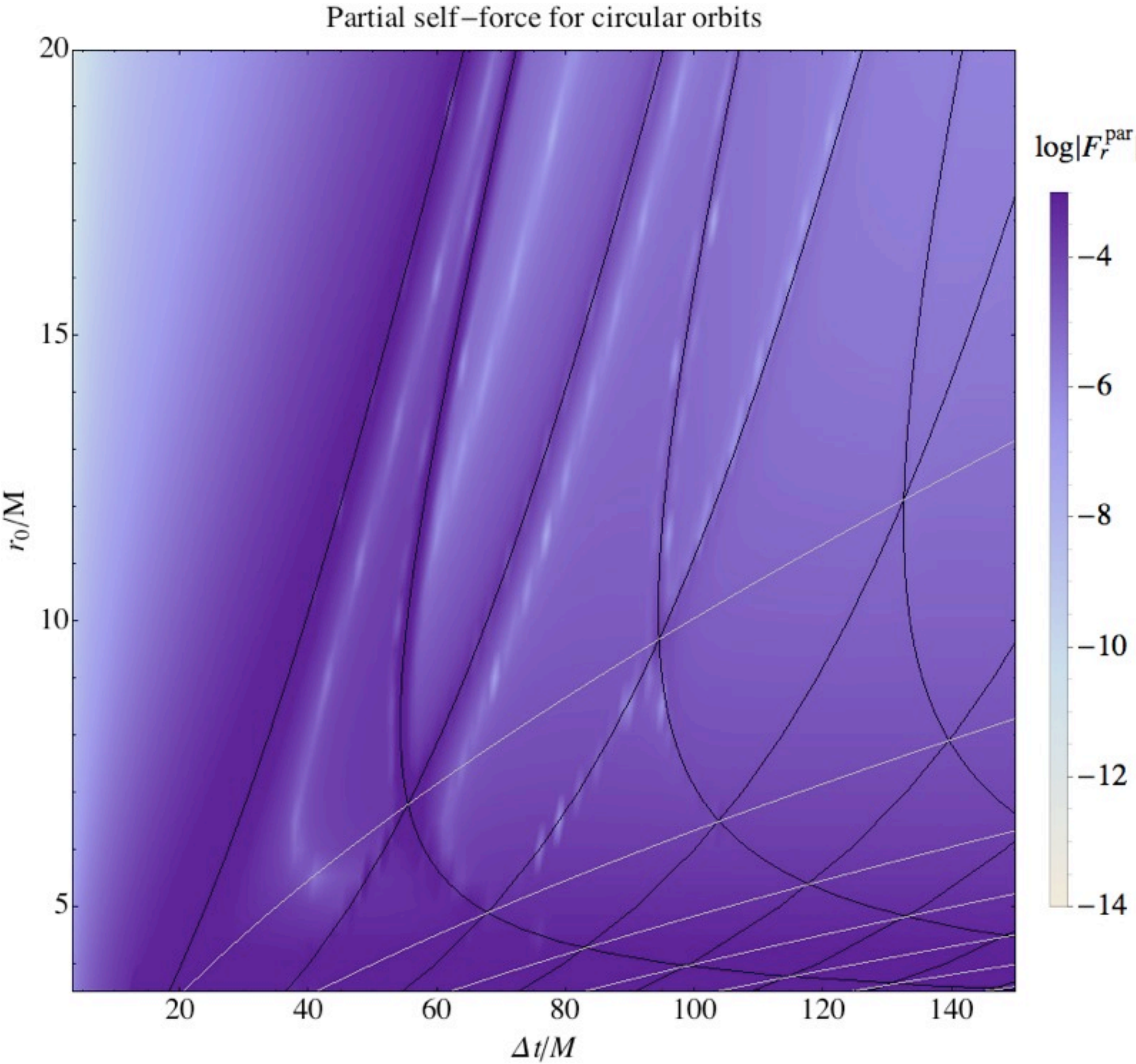


Computing the self-force

Computing the self-force

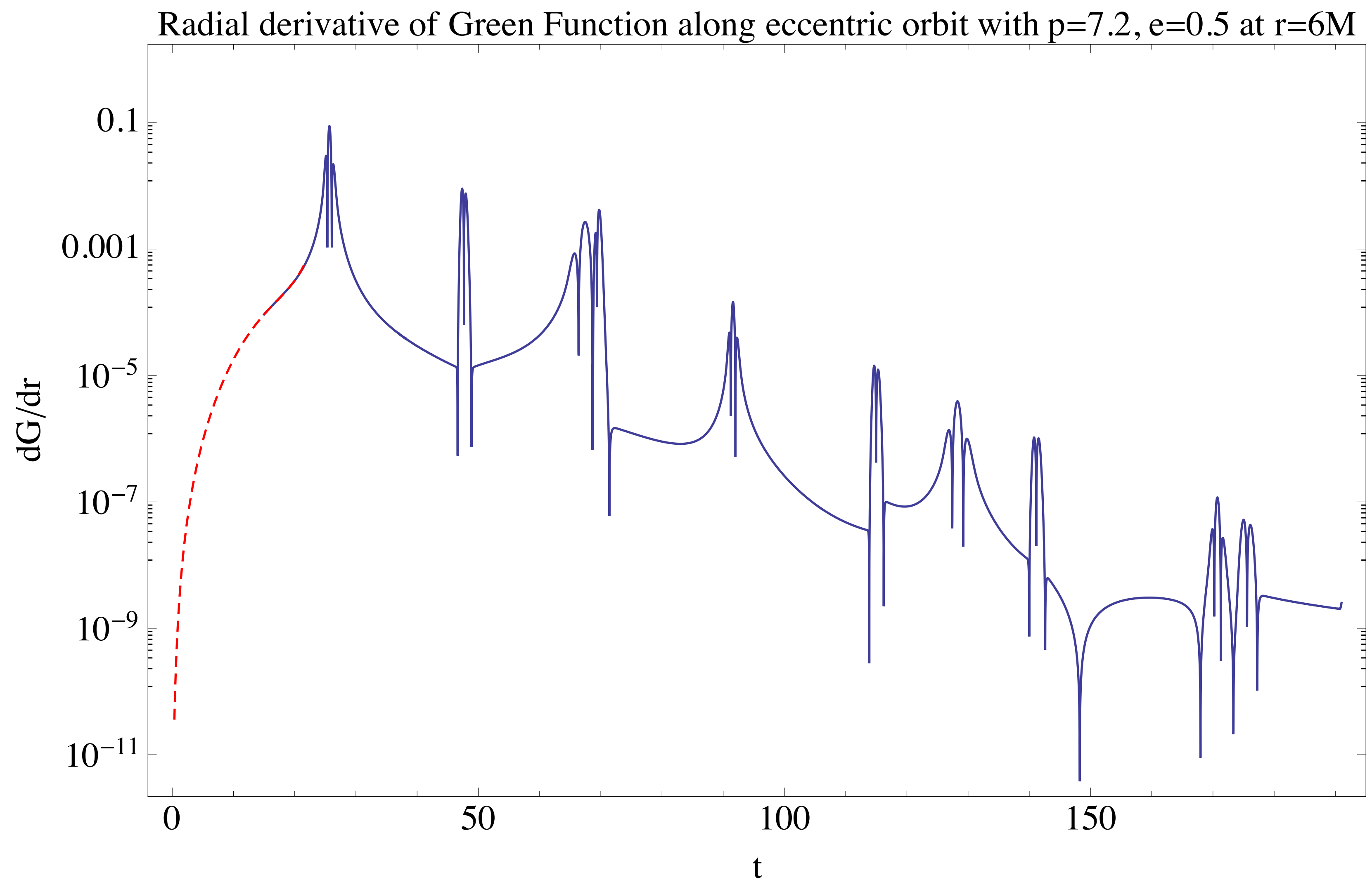
- Integrate matched Green function to get regularized field / self-force




Partial self−force for circular orbits
20
15
10
5
r_0/M
20
40
60
80
100
120
140
$\Delta t/M$
$\log|F_r^{\text{par}}|$
−4
−6
−8
−10
−12
−14

Generic orbits

- Green function approach works equally well with all types of orbit.



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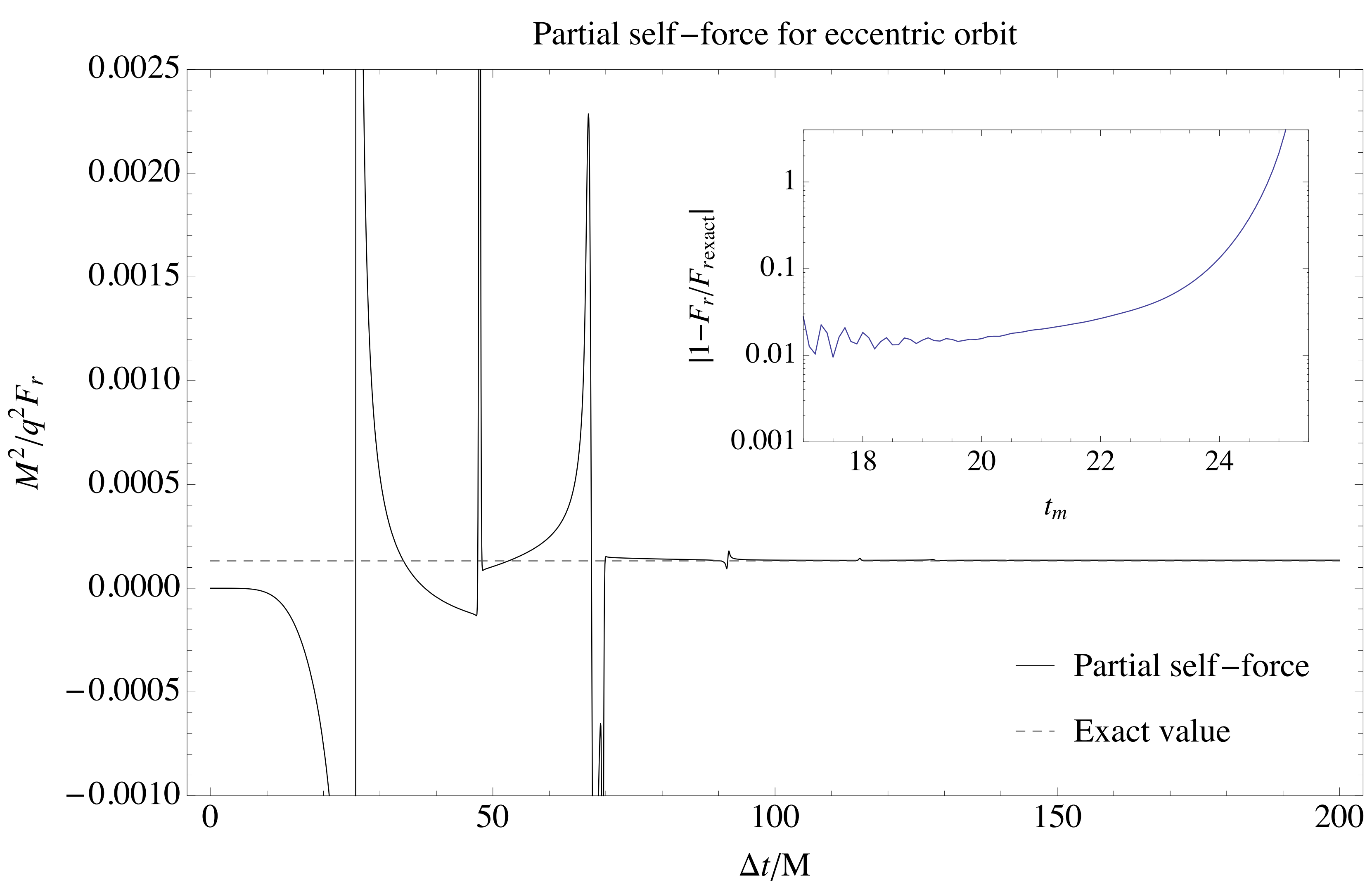

Conclusions

- Advantages:
  - Compute the Green function once, get the self-force for all orbits (including unbound, highly-eccentric, zoom-whirl, ultra-relativistic, which are difficult or inaccessible with existing methods).
  - Avoids numerical cancellation by directly computing regularized field.
  - May yield geometric insight.
  - Green function can be applied to other problems.
- Disadvantages:
  - Computing the Green function can be hard.
  - Have to compute the Green function for all pairs of points x and x'.
  - Not naturally suited to self-consistent evolution.
  - Second order not so well understood.

Conclusions and prospects

- Green functions are a flexible approach to self-force calculations.
- Compute Green function once, get all orbits through that base point.
- Need a separate calculation for each point on the orbit.
- Gives insight into how much of the past matters for the self-force.
- Interesting orbits not accessible by other means?
- Schwarzschild case now complete [arXiv:1306.0884].
- Application to Kerr spacetime.
- Extension to gravitational case.
- Self-force as a test of alternative theories of gravity?
- Other applications beyond self-force.