Brown-Henneaux's Canonical Approach to Topologically Massive Gravity

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(+ work in progress, T. Nishinaka)

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1. Introduction

Three dimensional gravity with negative cosmological constant has been one of the interesting testing grounds to uncover quantum natures of gravity.

The action is given by

$$S_{\mathrm{EH}} = \frac{1}{16\pi G_N}\int d^3x\sqrt{-G}\left(R + \frac{2}{\ell^2}\right)$$

The vacuum solution is given by global AdS_3 geometry.

$$ds^2 = -\left(1+\frac{r^2}{\ell^2}\right)dt^2 + \left(1+\frac{r^2}{\ell^2}\right)^{-1}dr^2 + r^2 d\phi^2$$

This theory also contains a black hole solution (BTZ black hole) which has mass and angular momentum.

Banados, Teitelboim, Zanelli

$$ds^2 = -N^2 dt^2 + N^{-2}dr^2 + r^2(d\phi + N^\phi dt)^2,$$

$$N^2 = \left(\frac{r}{\ell}\right)^2 + \left(\frac{4G_N j}{r}\right)^2 - 8G_N m, \quad N^\phi = \frac{4G_N j}{r^2}$$

BTZ black hole has inner and outer horizons :

$$r_\pm = \sqrt{2G_N\ell(\ell m + j)} \pm \sqrt{2G_N\ell(\ell m - j)} \qquad \ell m \geq j$$

By using the area formula, the entropy of the BTZ black hole is evaluated as

$$S = \frac{\pi}{2G_N}\left(\sqrt{2G_N\ell(\ell m + j)} + \sqrt{2G_N\ell(\ell m - j)}\right)$$

This is a thermodynamic entropy. Then it is natural to ask whether we can derive the above quantity from the statistical viewpoint.

The answer is yes. Brown and Henneaux showed that there exist Virasoro algebras at the boundary $r \to \infty$. From their prescription, the central charges for left and right moving modes are evaluated as

$$c_L = c_R = \frac{3\ell}{2G_N}$$

Brown, Henneaux

The statistical entropy can be calculated by Cardy formula, and the result coincides with the thermodynamic one.

The main purpose of this talk is to generalize Brown-Henneaux's canonical approach to topologically massive gravity (TMG).

$$\mathcal{S}_{\mathrm{TMG}} = \frac{1}{16\pi G_N}\int d^3x\,(\mathcal{L}_{\mathrm{EH}} + \mathcal{L}_{\mathrm{CS}})$$

Deser, Jackiw

$$\mathcal{L}_{\mathrm{EH}} = \sqrt{-G}\Big(R + \frac{2}{\ell^2}\Big)$$

$$\mathcal{L}_{\mathrm{CS}} = \frac{\beta}{2}\sqrt{-G}\epsilon^{IJK}\Big(\Gamma^P{}_{IQ}\partial_J\Gamma^Q{}_{KP} + \frac{2}{3}\Gamma^P{}_{IQ}\Gamma^Q{}_{JR}\Gamma^R{}_{KP}\Big)$$

This theory is important in the sense that three dimensional theories obtained by compactifying string theory or M-theory always contain Chern-Simons term.

Global AdS_3 geometry and BTZ black hole still become the solutions, even if the Chern-Simons term exists.

As we will see later, from the canonical approach it is possible to construct Virasoro algebras for left and right moving modes with

$$c_L \neq c_R$$

The plan of this talk :

1. Introduction
2. General Entropy Formula for BTZ Black Holes
3. Hamiltonian Formalism and Virasoro Algebras
4. Generalization to Topologically Massive Gravity
5. Central Charges with All Higher Derivative Corrections
6. Realization in M-theory: M5 System
7. Summary and Discussion
8. Holographic RG-flow in TMG (Work in Progress)

2. General Entropy Formula for BTZ Black Holes

In three dimensional theory, Riemann tensor can be expressed by Ricci tensor and scalar curvature.

Then generalized TMG action is written in the form

$$S = \frac{1}{16\pi G_N}\int d^3x\sqrt{-G}\left[f(R_{IJ}, G_{IJ}) + \frac{2}{\ell_0^2}\right] + \frac{1}{16\pi G_N}\int d^3x \mathcal{L}_{\mathrm{CS}}$$

$$f = R + aR^{IJ}R_{IJ} + bR^2 + \cdots$$

Equations of motion are expressed as

$$\frac{1}{2}G^{IJ}\left(f + \frac{2}{\ell_0^2}\right) + \frac{\partial f}{\partial G_{IJ}} = \underline{-T^{IJ}} + \beta\epsilon^{KL(I}\mathcal{D}_K R_L^{J)}$$

Contains covariant derivatives of Ricci tensor

Because the right hand side contains the covariant derivative of Ricci tensor, geometries which satisfy

$$\frac{1}{2}G^{IJ}\left(R + \frac{2}{\ell^2}\right) - R^{IJ} = 0$$

become solutions by adjusting $\ell = \ell(\ell_0)$ ⇒ Global AdS_3, BTZ still exist

Let us evaluate the black hole entropy.

It is known that the area law is modified by Wald's entropy formula for general covariant theories which include higher derivative terms.

Wald

Since the Chern-Simons part is non-covariant, additional contribution appears in the entropy formula.

Solodukhin; Tachikawa

$$
\begin{aligned}
S &= -\frac{1}{8G_N}\oint_{r_+} d\phi\sqrt{G_{\phi\phi}}\frac{\partial f}{\partial R_{IK}}G^{JL}\varepsilon_{IJ}\varepsilon_{KL} + \frac{\beta}{4G_N}\oint_{r_+} d\phi\,\varepsilon^{JI}\Gamma_{IJ\phi} \\
&= \frac{1}{4G_N}\Omega\oint_{r_+} d\phi\sqrt{G_{\phi\phi}} + \frac{\beta}{4G_N}\oint_{r_+} d\phi\frac{r_+r_-}{\ell r} \\
&= \frac{\pi\Omega}{2G_N}r_+ + \frac{\pi\beta}{2G_N\ell}r_-
\end{aligned}
$$

$$\Omega = \frac{1}{3}G_{IJ}\frac{\partial f}{\partial R_{IJ}}$$

Substituting $r_\pm = \sqrt{2G_N\ell(\ell m + j)} \pm \sqrt{2G_N\ell(\ell m - j)}$

The entropy becomes

$$S = \frac{\pi}{2G_N}\left\{\left(\Omega + \frac{\beta}{\ell}\right)\sqrt{2G_N\ell\,(m\ell + j)} + \left(\Omega - \frac{\beta}{\ell}\right)\sqrt{2G_N\ell\,(m\ell - j)}\right\}$$

This is a thermodynamic entropy. we need some statistical explanation.

3. Hamiltonian Formalism and Virasoro Algebras

Our goal is to evaluate the central charges of Virasoro algebras on the boundary in generalized TMG. To execute this calculation, we need Hamiltonian formalism of the system.

In this part, we review the essence of Hamiltonian formalism in pure gravity with negative cosmological constant.

Brown, Henneaux

First let us consider isometries of global AdS_3 or BTZ black hole at the boundary. From their line elements, we find following asymptotic behavior.

$$G_{tt} = -\frac{r^2}{\ell^2} + \mathcal{O}(1), \quad G_{tr} = \mathcal{O}(r^{-3}), \quad G_{t\phi} = \mathcal{O}(1)$$
$$G_{rr} = \frac{\ell^2}{r^2} + \mathcal{O}(r^{-4}), \quad G_{r\phi} = \mathcal{O}(r^{-3}), \quad G_{\phi\phi} = r^2 + \mathcal{O}(1)$$

This behavior is preserved under the coordinate transformations of

$$t \to t + \bar{\xi}^t, \quad r \to r + \bar{\xi}^r, \quad \phi \to \phi + \bar{\xi}^\phi$$

$$\bar{\xi}^t = \frac{\ell}{2} e^{inx^\pm}\left(1 - \frac{\ell^2 n^2}{2r^2}\right), \quad \bar{\xi}^r = -i\frac{nr}{2} e^{inx^\pm}, \quad \bar{\xi}^\phi = \pm\frac{1}{2} e^{inx^\pm}\left(1 + \frac{\ell^2 n^2}{2r^2}\right)$$

$n \in \mathbf{Z}$

$x^\pm = t/\ell \pm \phi$

Then Killing vector fields $\xi_n^{\pm} \equiv \bar{\xi}^I \partial_I$ satisfy commutation relations of

$$[\xi_m^{\pm}, \xi_n^{\pm}] = -i\,(m-n)\,\xi_{m+n}^{\pm}\,, \qquad [\xi_m^{+}, \xi_n^{-}] = \mathcal{O}(r^{-4})$$

This result shows that the asymptotically AdS_3 spacetime is endowed with the 2D conformal symmetry on the boundary.

In order to evaluate the central extension of the Virasoro algebras, we have to do following procedures.

A) Hamiltonian formalism.

B) Calculate the variation of the Hamiltonian, and add surface term to obtain correct equations of motion.

C) From this surface term, we obtain global charge. Possible to evaluate central charges.

A) Hamiltonian formalism.

We introduce the (1+2)-dimensional ADM decomposition of the metric.

$$G_{IJ} = \begin{pmatrix} -N^2 + N_k N^k & N_j \\ N_i & g_{ij} \end{pmatrix}$$

Then the Einstein-Hilbert Lagrangian with negative cosmological constant becomes

$$\mathcal{L}_{\mathrm{EH}} = \sqrt{g} N \left(r + \frac{2}{\ell^2} + K^{ij} K_{ij} - K^2 \right)$$

$$K_{ij} = \frac{1}{2N} \left(\dot{g}_{ij} - \mathcal{D}_i N_j - \mathcal{D}_j N_i \right)$$

$$K = g^{ij} K_{ij}$$

Canonical variable conjugate to g_{ij} is defined as usual and given by

$$\pi^{ij} = \sqrt{g} (K^{ij} - g^{ij} K)$$

g_{ij}, π^{ij} : canonical variables

N, N^i : auxiliary fields

Then Hamiltonian is constructed as

$$\mathcal{H}_{\mathrm{EH}} = \pi^{ij} \dot{g}_{ij} - \mathcal{L}_{\mathrm{EH}}$$

$$= N \{ -\sqrt{g}(r + 2/\ell^2) + g^{-1/2} (\pi^{ij} \pi_{ij} - \pi^2) \} + N^i \{ -2\sqrt{g} D_j (g^{-1/2} \pi_i{}^j) \}$$

B) Calculate the variation of the Hamiltonian. Add surface term to obtain correct equations of motion.

Variations with respect to N and N^i give initial value constraints.

$$\delta\mathcal{H}_{\rm EH} = \delta N\{-\sqrt{g}(r+\ell^2/2) + g^{-1/2}(\pi^{ij}\pi_{ij} - \pi^2)\} + \delta N^i\{-2\sqrt{g}\mathcal{D}_j(g^{-1/2}\pi_i{}^j)\}$$

Variations with respect canonical variables become

$$\begin{aligned}\delta\mathcal{H}_{\rm EH} = &+\{2g^{-1/2}N(\pi_{ij} - g_{ij}\pi) + 2\mathcal{D}_{(i}N_{j)}\}\delta\pi^{ij} \\ &+\{\sqrt{g}N(r^{ij} - \tfrac{1}{2}g^{ij}r) + 2g^{-1/2}N(\pi^{ik}\pi^j{}_k - \pi\pi^{ij}) - \tfrac{1}{2}g^{-1/2}Ng^{ij}(\pi^{kl}\pi_{kl} - \pi^2) \\ &\quad -\sqrt{g}(\mathcal{D}^i\mathcal{D}^jN - g^{ij}\mathcal{D}_k\mathcal{D}^kN) + 2\pi^{k(i}\mathcal{D}_kN^{j)} - \sqrt{g}\mathcal{D}_k(g^{-1/2}N^k\pi^{ij})\}\delta g_{ij} \\ &-\partial_l\{\sqrt{g}S^{ijkl}(N\mathcal{D}_k\delta g_{ij} - \mathcal{D}_kN\delta g_{ij}) + (2N^i\pi^{jl} - N^l\pi^{ij})\delta g_{ij} + 2N_i\delta\pi^{il}\}\end{aligned}$$

$$S^{ijkl} = \frac{1}{2}\left(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl}\right)$$

Correct equations of motion can be derived, only when the surface term is added to the original Hamiltonian. The variation of the surface term should cancel unwanted contributions.

Regge, Teitelboim

$$\delta Q[\xi] = \int d\phi\,\{\sqrt{g}S^{ijkr}(\xi^0\mathcal{D}_k\delta g_{ij} - \mathcal{D}_k\xi^0\delta g_{ij}) + (2\xi^i\pi^{jr} - \xi^r\pi^{ij})\delta g_{ij} + 2\xi_i\delta\pi^{ir}\}$$

$$(\xi^0, \xi^r, \xi^\phi) = (N\bar{\xi}^t, \bar{\xi}^r + N^r\bar{\xi}^t, \bar{\xi}^\phi + N^\phi\bar{\xi}^t)$$

$(\bar{\xi}^t, \bar{\xi}^r, \bar{\xi}^\phi) = (1, 0, 0)$: time translation

C) From this surface term, we obtain global charge. Possible to evaluate central charges.

The generator of isometry (Hamiltonian) consists of the constraint part together with appropriate surface term.

$$\mathcal{H}[\xi] = \int d^2x \left(\xi^0 \mathcal{H}_0 + \xi^i \mathcal{H}_i\right) + Q[\xi]$$

Algebraic structure of symmetric transformation group is given by the Poisson bracket of generators.

$$\{\mathcal{H}[\xi], \mathcal{H}[\eta]\}_{\mathrm{P}} = \mathcal{H}\big[[\xi, \eta]\big] + K[\xi, \eta]$$

The last term gives the central extension of the algebra and becomes

$$\begin{aligned} K[\xi,\eta] &= \delta_\eta Q[\xi] \\ &= \int d\phi \left\{\sqrt{g} S^{ijkr} (\xi^0 \mathcal{D}_k \delta_\eta g_{ij} - \mathcal{D}_k \xi^0 \delta_\eta g_{ij}) + 2\xi^i \pi^{jr} \delta_\eta g_{ij} + 2\xi_i \delta_\eta \pi^{ir} - \xi^r \pi^{ij} \delta_\eta g_{ij}\right\} \end{aligned}$$

This can be evaluated by substituting asymptotic values of global AdS_3. Finally we obtain

$$c_L = c_R = \frac{3\ell}{2G_N}$$

Before closing this section, let us consider Virasoro algebras of diffeomorphism invariant action

$$\mathcal{S} = \frac{1}{16\pi G_N} \int d^3x \sqrt{-G} \left\{ f(R_{IJ}, G_{IJ}) + \frac{2}{\ell_0^2} \right\}$$

The important point is that the Lagrangian constructed out of the metric and the Ricci tensor is equivalent to the Einstein-Hilbert Lagrangian with matter fields after the frame transformation.

Magnano, Ferraris, Francaviglia

The metric in the Einstein frame is given by scaling the original metric (after substituting AdS or BTZ solutions)

$$\tilde{G}_{IJ} = \Omega^2 G_{IJ} \qquad \Omega = \frac{1}{3} G_{IJ} \frac{\partial f}{\partial R_{IJ}}$$

Then canonical variables scale as

$$\tilde{g}_{ij} = \Omega^2 g_{ij}, \quad \tilde{N} = \Omega N, \quad \tilde{N}^i = N^i, \quad \tilde{\pi}^{ij} = \Omega^{-1} \pi^{ij}$$

Finally mass, angular momentum and central charges are evaluated like

$$M = \Omega m, \quad J = \Omega j, \quad c_L = c_R = \Omega \frac{3\ell}{2G_N}$$

Saida, Soda

4. Generalization to Topologically Massive Gravity

Let us apply the arguments so far to TMG. We will confirm that mass and angular momentum of the BTZ black hole and central charges of the boundary CFT are all modified in TMG.

The action of TMG is given by Einstein-Hilbert term with negative cosmological constant and gravitational Chern-Simons term. ADM decomposition of gravitational Chern-Simons term is given by

$$\sqrt{-G}\epsilon^{IJK}\left(\Gamma^P{}_{IQ}{}_J\Gamma^Q{}_{KP}+\frac{2}{3}\Gamma^P{}_{IQ}\Gamma^Q{}_{JR}\Gamma^R{}_{KP}\right)$$

$$\cong 2\sqrt{g}\epsilon^{mn}\dot{K}_{mk}K_n{}^k+\sqrt{g}N\{4\epsilon^{mn}\mathcal{D}_k\mathcal{D}_nK_m{}^k-2A^{kl}K_{kl}\}$$

$$+\sqrt{g}N^i\{-4\epsilon^{mn}K_i{}^l\mathcal{D}_nK_{ml}-2\epsilon^{mn}\mathcal{D}_k(K_{ni}K_m{}^k)+\epsilon_{ij}\partial^jr+2\mathcal{D}_kA_i{}^k\}$$

$$A_{ij}=A_{ij}(\gamma)$$

This contains third derivative with respect to time. It is known that the canonical formalism of such system is done by using Ostrogradsky method in which Lagrange multiplier is introduced.

Ostrogradsky

$$\mathcal{L}(g,\dot{g},\ddot{g})\ \longrightarrow\ \mathcal{L}^*(g,\dot{g},K,\dot{K},v)=\mathcal{L}(g,K,\dot{K})+v(\dot{g}-K)$$

It is convenient to choose K_{ij} as an independent variable.

Buchbinder, Lyahovich, Krykhtin

Hamiltonian is constructed in an usual manner. Defining π^{ij} and Π^{ij} as momenta conjugate to g_{ij} and K_{ij} respectively, we obtain

$$\mathcal{H}_{\mathsf{TMG}} = \pi^{ij}\dot{g}_{ij} + \Pi^{ij}\dot{K}_{ij} - \mathcal{L}_{\mathsf{TMG}} + f_{ij}\Big(\Pi^{ij} - \beta\sqrt{g}\epsilon^{ik}K_k{}^j\Big)$$
$$\cong \sqrt{g}N\Big\{-r - \frac{2}{\ell^2} - K^{kl}K_{kl} + K^2 - 2\beta\epsilon^{mn}\mathcal{D}_k\mathcal{D}_nK_m{}^k + \Big(2g^{-\frac{1}{2}}\pi^{kl} + \beta A^{kl}\Big)K_{kl}\Big\}$$
$$+ \sqrt{g}N^i\Big\{2\beta\epsilon^{mn}K_i{}^l\mathcal{D}_nK_{ml} + \beta\epsilon^{mn}\mathcal{D}_k(K_{ni}K_m{}^k) - \frac{1}{2}\beta\epsilon_{ij}\partial^j r - \mathcal{D}_j\Big(2g^{-\frac{1}{2}}\pi_i{}^j + \beta A_i{}^j\Big)\Big\}$$
$$+ f_{ij}\Big(\Pi^{ij} - \beta\sqrt{g}\epsilon^{ik}K_k{}^j\Big)$$

Then in order to derive correct equations of motion, it is necessary to add surface term $Q[\xi]$ to the Hamiltonian.

$$\delta Q[\xi] = \int d\phi\Big[\sqrt{g}S^{ijkr}\Big(\xi^0\mathcal{D}_k\delta g_{ij} - \mathcal{D}_k\xi^0\delta g_{ij}\Big) + \xi^i(2\pi^{jr} + \beta g^{\frac{1}{2}}A^{jr})\delta g_{ij} - \frac{1}{2}\xi^r(2\pi^{ij} + \beta g^{\frac{1}{2}}A^{ij})\delta g_{ij}$$
$$+ \xi_i\delta(2\pi^{ir} + \beta g^{\frac{1}{2}}A^{ir}) - 2\beta\sqrt{g}\epsilon^{mr}\mathcal{D}_k\xi^0 g^{kl}\delta K_{ml} + 2\beta\sqrt{g}\epsilon^{mn}\xi^0\mathcal{D}_n(g^{rl}\delta K_{ml})$$
$$+ \frac{1}{2}\beta\sqrt{g}S^{ijkr}\Big((\epsilon^{mn}\partial_m\xi_n)\mathcal{D}_k\delta g_{ij} - \mathcal{D}_k(\epsilon^{mn}\partial_m\xi_n)\delta g_{ij}\Big)$$
$$- 2\beta\sqrt{g}\epsilon^{mr}\xi^iK_i{}^l\delta K_{ml} - \beta\sqrt{g}\epsilon^{mn}\xi^i(\delta K_{ni}K_m{}^r + K_{ni}\delta K_m{}^r) - 2\beta\sqrt{g}\epsilon^{mn}g^{rl}\xi^0K_{mo}\delta\gamma^o{}_{nl}$$
$$- 2\beta\sqrt{g}\epsilon^{mn}g^{kl}g^{op}\Big\{-\mathcal{D}_k\xi^0K_{mo}T^{ijr}_{nlp} + \xi^0\mathcal{D}_oK_{ml}T^{ijr}_{knp} + 2\xi^0\mathcal{D}_nK_{o(l}T^{ijr}_{m)kp}\Big\}\delta g_{ij}$$
$$+ 2\beta\sqrt{g}\epsilon^{mn}\xi^o\Big\{K_o{}^pK_{mk}g^{kl}T^{ijr}_{npl} + g^{qk}g^{lp}(K_{mk}K_{l(o}T^{ijr}_{n)qp} + K_{no}K_{l(k}T^{ijr}_{m)qp})\Big\}\delta g_{ij}$$
$$+ \frac{1}{2}\beta\sqrt{g}\epsilon^{mn}T^{xyk}_{mlo}g^{op}\Big\{\mathcal{D}_ku_{xy}g^{lq}T^{ijr}_{npq} - u_{xy}\gamma^q{}_{np}g^{ls}T^{ijr}_{kqs} + 2u_{xy}\gamma^l{}_{q(p}g^{qs}T^{ijr}_{n)ks}\Big\}\delta g_{ij}$$
$$- \frac{1}{2}\beta\sqrt{g}\epsilon^{mn}T^{ijr}_{mlo}u_{ij}g^{op}\delta\gamma^l{}_{np} + \frac{1}{2}\beta\sqrt{g}\epsilon^{mr}\xi_m\delta r\Big]$$

Now it is possible to evaluate mass and angular momentum of BTZ black holes, and central charges at the boundary.

Mass

$$
\begin{aligned}
(\xi^0, \xi^r, \xi^\phi) &= (N, N^r, N^\phi) \sim \left(\frac{r}{\ell},\ 0,\ \frac{4G_N j}{r^2}\right) \\
M &= \frac{1}{16\pi G_N}\delta Q[\xi] \\
&= \frac{1}{16\pi G_N}\oint_{r=\infty} d\phi \Big\{ 2\sqrt{g} S^{r\phi r\phi}(-\xi^0 \gamma^r{}_{\phi\phi}\delta g_{rr}) + 2\beta \mathcal{D}_k \xi^0 g^{kl}\delta K_{\phi l} \Big\} \\
&= m + \frac{\beta}{\ell^2} j
\end{aligned}
$$

Angular mom.

$$
\begin{aligned}
(\xi^0, \xi^r, \xi^\phi) &= (0, 0, 1) \\
J &= \frac{1}{16\pi G_N}\delta Q[\xi] \\
&= \frac{1}{16\pi G_N}\oint_{r=\infty} d\phi \Big\{ \xi_i \delta(2\pi^{ir} + \beta g^{\frac{1}{2}} A^{ir}) + \beta\sqrt{g} S^{r\phi r\phi}(-\epsilon^{mn}\partial_m \xi_n \gamma^r{}_{\phi\phi}\delta g_{rr}) \Big\} \\
&= j + \beta m
\end{aligned}
$$

Central charges

$$(\xi^0, \xi^r, \xi^\phi) \sim \left(\frac{r}{2}e^{inx^\pm}, -i\frac{nr}{2}e^{inx^\pm}, \pm\frac{1}{2}e^{inx^\pm}\right)$$

$$\begin{aligned}\frac{1}{16\pi G_N}\delta_{\eta=\xi_n^+}Q[\xi=\xi_m^+] &= \frac{1}{16\pi G_N}\oint_{r=\infty} d\phi\Big\{\tfrac{1}{\ell}\Big(\tfrac{1}{r}\xi^0+\partial_r\xi^0\Big)\delta_\eta g_{\phi\phi}+\tfrac{r^3}{\ell^3}\xi^0\delta_\eta g_{rr}+\xi_\phi\delta_\eta(2\pi^{\phi r}+\beta g^{\frac{1}{2}}A^{\phi r})\Big\}\\ &\quad+\frac{\beta}{16\pi G_N}\oint_{r=\infty} d\phi\left\{\tfrac{1}{\ell^2}\Big(\tfrac{1}{r}\xi^0+\partial_r\xi^0\Big)\delta_\eta g_{\phi\phi}+\tfrac{r^3}{\ell^4}\xi^0\delta_\eta g_{rr}+2\partial_r\xi^0 g^{rl}\delta_\eta K_{\phi l}\right\}\\ &= -\frac{i}{12}\frac{3\ell}{2G_N}\left(1+\frac{\beta}{\ell}\right)m(m^2-1)\delta_{m,-n}\end{aligned}$$

$$\begin{aligned}\frac{1}{16\pi G_N}\delta_{\eta=\xi_n^-}Q[\xi=\xi_m^-] &= \frac{1}{16\pi G_N}\oint_{r=\infty} d\phi\Big\{\tfrac{1}{\ell}\Big(\tfrac{1}{r}\xi^0+\partial_r\xi^0\Big)\delta_\eta g_{\phi\phi}+\tfrac{r^3}{\ell^3}\xi^0\delta_\eta g_{rr}+\xi_\phi\delta_\eta(2\pi^{\phi r}+\beta g^{\frac{1}{2}}A^{\phi r})\Big\}\\ &\quad+\frac{\beta}{16\pi G_N}\oint_{r=\infty} d\phi\left\{-\tfrac{1}{\ell^2}\Big(\tfrac{1}{r}\xi^0+\partial_r\xi^0\Big)\delta_\eta g_{\phi\phi}-\tfrac{r^3}{\ell^4}\xi^0\delta_\eta g_{rr}+2\partial_r\xi^0 g^{rl}\delta_\eta K_{\phi l}\right\}\\ &= -\frac{i}{12}\frac{3\ell}{2G_N}\left(1-\frac{\beta}{\ell}\right)m(m^2-1)\delta_{m,-n}\end{aligned}$$

Thus we obtain left-right asymmetric central charges.

$$\begin{aligned}c_L &= \frac{3\ell}{2G_N}\left(1+\frac{\beta}{\ell}\right),\\ c_R &= \frac{3\ell}{2G_N}\left(1-\frac{\beta}{\ell}\right)\end{aligned}$$

5. Central Charges with All Higher Derivative Corrections

We have established the canonical formalism and have gotten the Virasoro central charges for TMG. We generalize these results in order to encompass general cases of higher derivative gravity.

$$S = \frac{1}{16\pi G_N}\int d^3x\sqrt{-G}\left[f(R_{IJ}, G_{IJ}) + \frac{2}{\ell_0^2}\right] + \frac{1}{16\pi G_N}\int d^3x \mathcal{L}_{\rm CS}$$

This theory contains global AdS_3 and BTZ black holes as solutions. Mass and angular momentum for BTZ black hole become

$$M = \Omega m + \frac{\beta}{\ell^2} j, \quad J = \Omega j + \beta m$$

CFT_2 exists at the boundary and central charges take following left-right asymmetric forms.

$$c_L = \frac{3\ell}{2G_N}\left(\Omega + \frac{\beta}{\ell}\right), \quad c_R = \frac{3\ell}{2G_N}\left(\Omega - \frac{\beta}{\ell}\right)$$

Note that $L_0^{\pm}$ correspond to the isometries $\xi_0^{\pm} = \partial_{\pm} = \frac{1}{2}(\ell\partial_t \pm \partial_\phi)$.
Therefore we obtain

$$L_0^{+} = \frac{1}{2}(M\ell + J) = \left(\Omega + \frac{\beta}{\ell}\right)\frac{1}{2}(m\ell + j)$$
$$L_0^{-} = \frac{1}{2}(M\ell - J) = \left(\Omega - \frac{\beta}{\ell}\right)\frac{1}{2}(m\ell - j)$$

From Cardy's formula for counting the states in CFT, we obtain the statistical entropy for BTZ black holes.

$$S = 2\pi\sqrt{\frac{1}{6}c_L L_0^{+}} + 2\pi\sqrt{\frac{1}{6}c_R L_0^{-}}$$
$$= \frac{\pi}{2G_N}\left(\Omega + \frac{\beta}{\ell}\right)\sqrt{2G_N\ell\,(m\ell + j)} + \frac{\pi}{2G_N}\left(\Omega - \frac{\beta}{\ell}\right)\sqrt{2G_N\ell\,(m\ell - j)}$$

This agrees with the previous thermodynamic entropy formula. For BTZ black holes capturing the contributions of all higher derivative corrections, we have thus proven the agreement between the macroscopic entropy and the Cardy's entropy of microstate counting.

6. Realization in M-theory: M5 System

The 3D theory is usually embedded in higher dimensional theories in the string theory context. An interesting example is embodied in M-theory, which is intriguing because the corresponding CFT is understood clearly.

BTZ $\times S^2 \times$ CY3 $\sim$ M5-branes wrapping on 4-cycles in CY3.

↓ Hanaki, Ohashi, Tachikawa; Gupta, Sen

Generalized TMG + moduli scalars

we solved for moduli scalars.

$$\ell = \frac{p^3}{4\pi}\left(1 + \frac{37}{288}\frac{c_{2I}p^I}{p^3}\right)$$

$$\Omega = 1 - \frac{c_{2I}p^I}{288p^3}$$

$$\beta = -c_{2I}p^I/96\pi$$

↔ consistent

↓ Maldacena, Strominger, Witten

$\mathcal{N} = (0,4)$ CFT2

$$c_L = 6p^3 + \frac{1}{2}c_{2I}p^I$$

$$c_R = 6p^3 + c_{2I}p^I$$

p^I: wrapping #

$p^3 \equiv \frac{1}{6}c_{IJK}p^Ip^Jp^K$

$c_{2I} = \frac{1}{8\pi^2}\int_{\mathrm{CY}_3} J_I \wedge \mathrm{tr}(R\wedge R)$

Represented by topological quantities.

7. Summary and Discussion

We have analyzed generalized topologically massive gravity using the conventional canonical formalism.

We defined the global charges so as to cancel the surface terms of the variation of the Hamiltonian. Virasoro algebras are derived at the boundary, and central charges are left-right asymmetry.

Mass and angular momentum of the BTZ black hole are also computed.

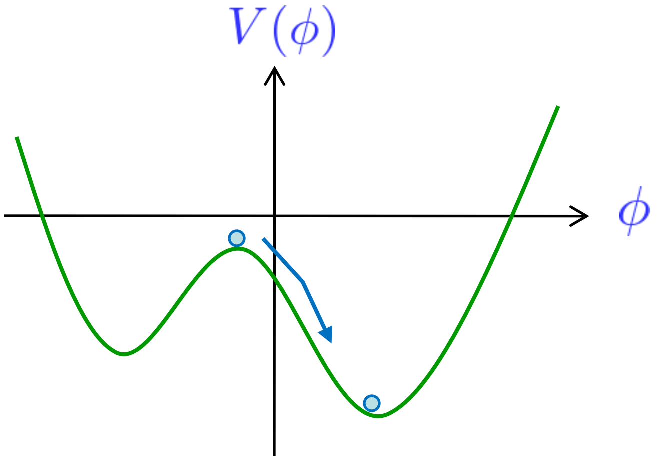
Thermodynamic entropy for BTZ black hole agrees with the statistical entropy on the boundary CFT. AdS/CFT confirmed.

In TMG, we considered two parameters ℓ and β which are continuous. Recent arguments show that these should be related.

⇨ Li, Song, Strominger

8. Holographic RG Flow in TMG (Work in Progress)

As an interesting generalization, we add a scalar field to TMG.
It interpolates between two AdS_3 vacua.



The radial coordinate is identified with the RG flow parameter of the boundary field theory.

De Boer, Verlinde, Verlinde
Fukuma, Matsuura, Sakai

Hamilton-Jacobi formulation

Is it possible to obtain c-functions in TMG?

Euclidean action of TMG with a scalar field is given by

$$S = \frac{1}{16\pi G_N} \int d^3x \sqrt{G} \Big\{ -R + \frac{1}{2} G^{IJ} \partial_I \phi \partial_J \phi + V(\phi) \\ - \frac{\beta}{2} \epsilon^{IJK} \Big(\Gamma^P{}_{IQ} \partial_J \Gamma^Q{}_{KP} + \frac{2}{3} \Gamma^P{}_{IQ} \Gamma^Q{}_{JR} \Gamma^R{}_{KP} \Big) \Big\}$$

Hamiltonian density is expressed as

$$\mathcal{H} = \pi^{ij} \dot{g}_{ij} + \Pi^{ij} \dot{K}_{ij} + \pi \dot{\phi} - L + f_{ij} \big(\Pi^{ij} + \beta \sqrt{g} \epsilon^{k(i} K_k{}^{j)} \big)$$

$$\cong \sqrt{g} N \Big\{ r - V(\phi) - K^{kl} K_{kl} + K^2 + 2\beta^{mn} \mathcal{D}_k \mathcal{D}_n K_m{}^k + \big(2g^{-\frac{1}{2}} \pi^{kl} - \beta A^{kl} \big) K_{kl} + \frac{1}{2} g^{-1} \pi^2 - \frac{1}{2} g^{ij} \partial_i \phi \partial_j \phi \Big\} \\ + \sqrt{g} N^i \Big\{ 2\beta^{mn} K_i{}^l \mathcal{D}_n K_{ml} + \beta^{mn} \mathcal{D}_k (K_{ni} K_m{}^k) + \frac{1}{2} \beta_{ij} \partial^j r - \mathcal{D}_j \big(2g^{-\frac{1}{2}} \pi_i{}^j - \beta A_i{}^j \big) + g^{-1/2} \partial_i \phi \, \pi \Big\} \\ + f_{ij} \big(\Pi^{ij} + \beta \sqrt{g}^{k(i} K_k{}^{j)} \big)$$

There are two constraints with $\Pi^{ij} = -\beta \sqrt{g}^{k(i} K_k{}^{j)}$

First constraint (Hamiltonian constraint) :

$$0 = r - V(\phi) - K^{kl}K_{kl} + K^2 + 2\beta^{mn}\mathcal{D}_k\mathcal{D}_n K_m{}^k + \left(2g^{-\frac{1}{2}}\pi^{kl} - \beta A^{kl}\right)K_{kl} + \frac{1}{2}g^{-1}\pi^2 - \frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi$$

$$= r - V(\phi) - \left(\frac{1}{\beta\sqrt{g}}\right)^2 \Pi_{ij}\Pi^{ij} + \tfrac{1}{2}K^2 - 2\mathcal{D}_i\mathcal{D}_j\left(\frac{1}{\sqrt{g}}\Pi^{ij}\right)$$

$$-\left(\frac{2}{\sqrt{g}}\pi^{ij} - \beta A^{ij}\right)\left(\frac{1}{\beta\sqrt{g}}\right)\epsilon_{ik}\Pi^k{}_j + \left(\frac{1}{\sqrt{g}}g_{ij}\pi^{ij} - \frac{1}{2}\beta g_{ij}A^{ij}\right)K + \frac{1}{2}g^{-1}\pi^2 - \frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi$$

H-J equation is obtained by setting $\pi^{ij} = \dfrac{\delta S}{\delta g_{ij}}, \ \Pi^{ij} = \dfrac{\delta S}{\delta K_{ij}}$

$$-\frac{1}{\beta^2}\left(\frac{1}{\sqrt{g}}\frac{\delta S}{\delta K_{ij}}\right)^2 - 2_{ij}\left(\frac{1}{\sqrt{g}}\frac{\delta S}{\delta K_{ij}}\right) - \frac{2}{\beta}g_{ij}\epsilon_{kl}\left(\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{ik}}\right)\left(\frac{1}{\sqrt{g}}\frac{\delta S}{\delta K_{jl}}\right) + Kg_{ij}\left(\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{ij}}\right) + \frac{1}{2}\left(\frac{1}{\sqrt{g}}\frac{\delta S}{\delta \phi}\right)^2$$

$$= V(\phi) - \frac{1}{2}K^2 - r + \frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi$$

This can be solved order by order with respect to *weight*, which is assigned as

$$w = 0: \;\; g_{ij},\; \phi,\; K_{ij},\; \Gamma[g, \phi, K]$$
$$w = 1: \;\; \partial_i$$
$$w = 2: \;\; R,\; \partial_i\phi\partial_j\phi$$

$$\frac{1}{16\pi G_N} S = \frac{1}{16\pi G_N} S_{\rm loc}^{w=-2} + \Gamma + \frac{1}{16\pi G_N} S_{\rm loc}^{w=2} + \cdots$$

$$L_{\rm loc}^{w=0} = \sqrt{g} W(\phi)$$

W=0 $$KW + \frac{1}{2}\left(\frac{W}{\phi}\right)^2 = V - \frac{1}{2}K^2$$

W=2 $$16\pi G_N K g_{ij}\left(\frac{1}{\sqrt{g}}\frac{\delta\Gamma}{\delta g_{ij}}\right) + 8\pi G_N \frac{\partial W}{\partial\phi}\left(\frac{1}{\sqrt{g}}\frac{\delta\Gamma}{\delta\phi}\right) = -r + \frac{1}{2}g^{ij}\partial_i\phi\partial_j\phi$$

$$\Rightarrow \quad c_L + c_R \sim \frac{1}{K}$$

Second constraint (Momentum constraint) :

$$0=\sqrt{g}\xi^{i}\left\{2\beta^{mn}K_{i}{}^{l}\mathcal{D}_{n}K_{ml}+\beta^{mn}\mathcal{D}_{k}(K_{ni}K_{m}{}^{k})+\frac{1}{2}\beta^{j}_{ij}r-_{j}\left(2g^{-\frac{1}{2}}\pi_{i}{}^{j}-\beta A_{i}{}^{j}\right)+g_{i}^{-1/2}\phi\pi\right\}$$

$$\cong\delta K_{ij}\frac{\delta S}{\delta K_{ij}}+\delta g_{ij}\frac{\delta S}{\delta g_{ij}}+\delta\phi\frac{\delta S}{\delta\phi}+\frac{1}{2}\beta\sqrt{g}\xi^{i}\epsilon_{ij}\partial^{j}r$$

Variation under general coordinate tr.

⇨ $c_L - c_R \sim \beta$

Combining the results, we obtain c-functions in the gravity side

$$c_L=\frac{-3}{2G_NK}\left(1-K\beta\right)$$

$$c_R=\frac{-3}{2G_NK}\left(1+K\beta\right)$$

K depends on the radial coordinate