The FZZ-duality conjecture - A Proof

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Refs.
[1] YH-Schomerus, JHEP10(2007)064 [0706.1030].
[2] YH-Schomerus, JHEP12(2007)100 [0711.0338].
[3] YH-Schomerus, arXiv:0805.3931 [hep-th].

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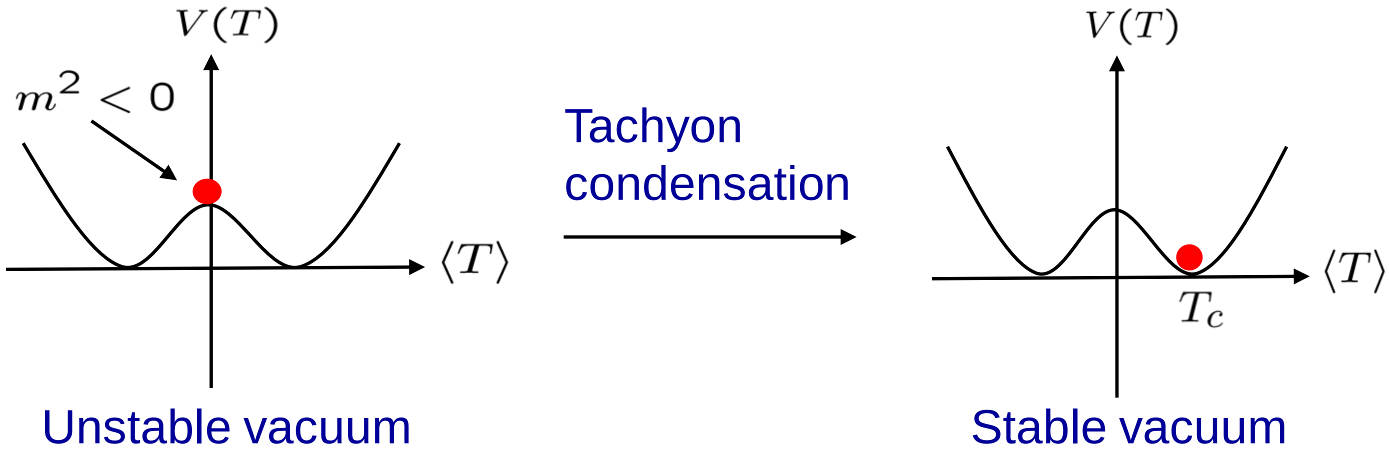
1. Introduction

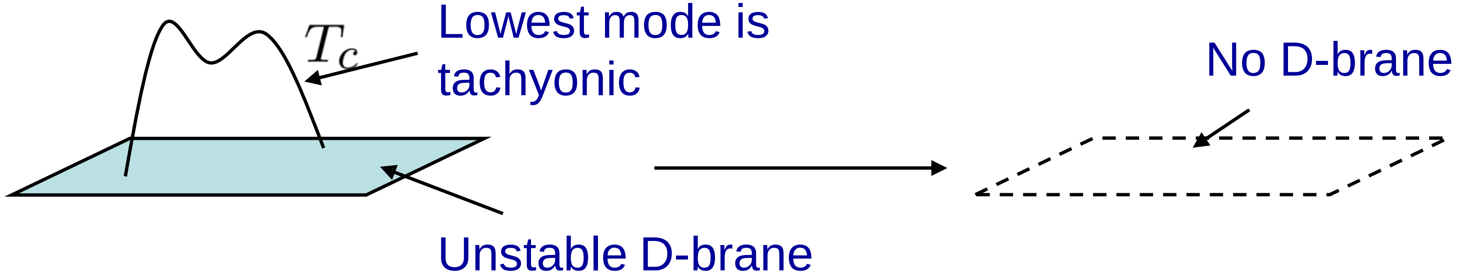
Closed string tachyon condensation and the FZZ duality

Tachyon condensation in string theory

- Tachyon condensation



- Open string tachyon condensation [Sen]



⇨ What happens if closed string tachyon is condensed?

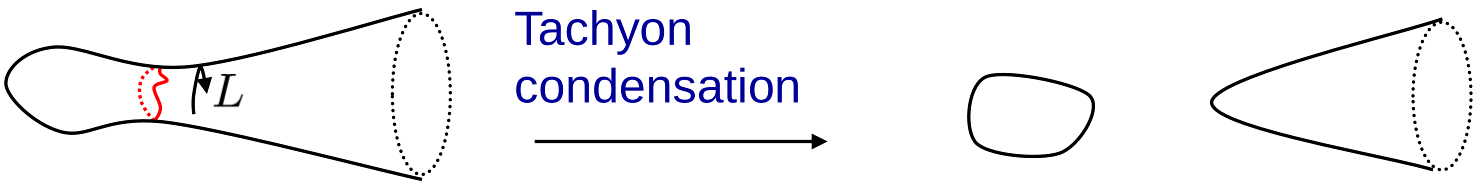
Closed string tachyon condensation

- Closed string tachyon condensation
  - Generically it is difficult to deal with since the background itself would change.

  ⇨ Localized tachyon condensation

  [Adams-Polchinski-Silverstein]

- Winding string tachyon



$$M^2 = -\frac{1}{l_s^2} + \frac{L^2}{l_s^4}$$ Tachyonic for $L < l_s$

  – Singularities may be removed by tachyon condensation.

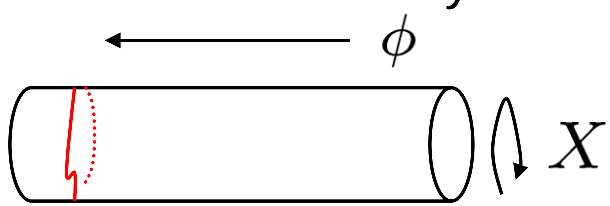
  [McGreevy-Silverstein, Horowitz-Silverstein]

  – Difficult to say something concrete, need to understand stringy effects.

The FZZ-duality conjecture

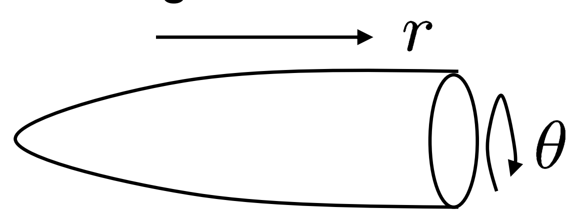
- The Fateev-Zamolodchikov^2 (FZZ) duality

Sine-Liouville theory



$$T = e^{\sqrt{k-2}\phi}\cos(\sqrt{k}\tilde{X})$$

Cigar model



$$ds^2 = k[dr^2 + \tanh^2 r d\theta^2]$$

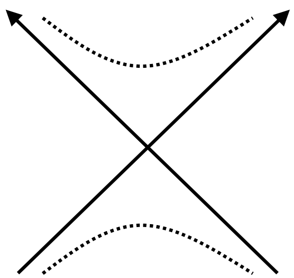
  - Condensation of winding string tachyon changes the geometry from cylinder to cigar.
  - Exact results in α' corrections
  - Weak-strong duality w.r.t. $k = R^2$ (like T-duality)

⇒ The aim of this talk is to proof this duality

A solvable model for 2d black hole

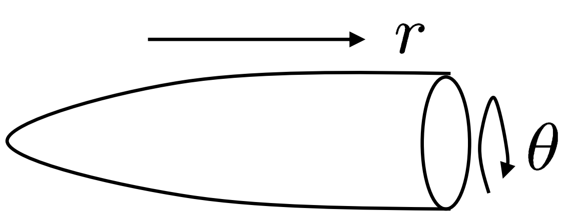
- 2d black hole
  - A solvable model is proposed by Witten, which can be constructed by gauging the H_3^+ model

☐ Lorentzian black hole



$$ds^2 = k\frac{dudv}{1-uv},$$
$$\Phi = \Phi_0 - \log(1-uv)$$

☐ Euclidean black hole (cigar)



$$ds^2 = k[dr^2 + \tanh^2 r d\theta^2]$$
$$\Phi = \Phi_0 - 2\log\cosh r$$

Strategy

- Strategy to proof the FZZ duality
  - The cigar model can be defined by the sum of H_3^+ model and free boson (and ghosts) [Dijkgraaf-Verlinde[2]]

$$S^{\text{cig}}[g,X] = S^{\text{WZNW}}[g] + \frac{1}{2\pi}\int d^2w\,\partial X\bar{\partial}X \ \ (+\text{FP ghosts})$$

The action of H_3^+ model (describing strings on AdS_3)

  - H_3^+- Liouville theory [Stoyanovski, Ribaut-Teschner]

N-pt. function of H_3^+ model	$\Leftrightarrow$	$(2N\text{-}2)$-pt. function of Liouville theory

$\Rightarrow$ We show that the combination of the two above facts leads to the FZZ duality.

Plan of this talk

1. Introduction
2. H_3^+- Liouville relation
   – Relation between H_3^+ model and Liouville theory
   – Path integral derivation
   – Few comments
3. The FZZ duality
   – The cigar model as a gauged WZNW model
   – The cigar – Liouville relation
   – The duality between the cigar and Sine-Liouville
4. Conclusion
5. Appendix

2. H_3^+- Liouville relation

The relation and a proof in path integral formulation

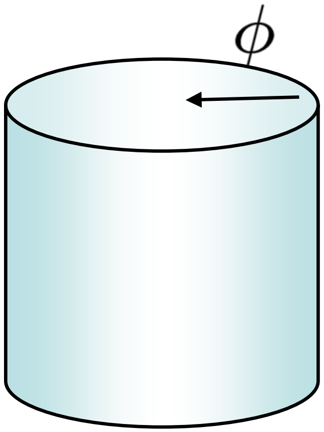
H_3^+-Liouville relation

- H_3^+-Liouville relation
  (Stoyanovsky-Ribault-Teschner relation)

N-pt. function of H_3^+ model
(string theory on Euclidian AdS_3)

⇕

(2*N*-2)-pt. function of Liouville theory
with (*N*-2) degenerate fields



- The results of our work
  - Path integral derivation of the relation
  - Generalizations and applications
    - On Riemann surface of a higher genus
    - WZNW model on supergroup
    - The FZZ duality from the coset construction

Liouville field theory

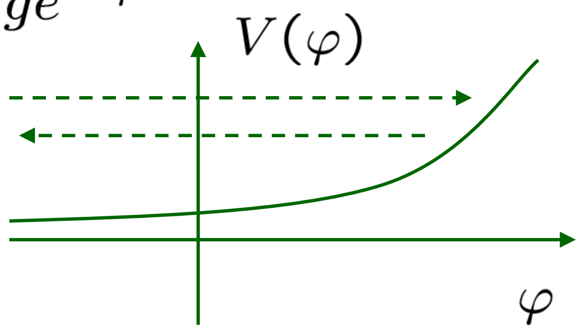
Liouville field theory is simplest model among non-compact CFTs and it may serve as a building block for general CFT.

- Action

$$S^L = S_{\text{free}} + S_{\text{LD}} + S_{\text{int}}$$

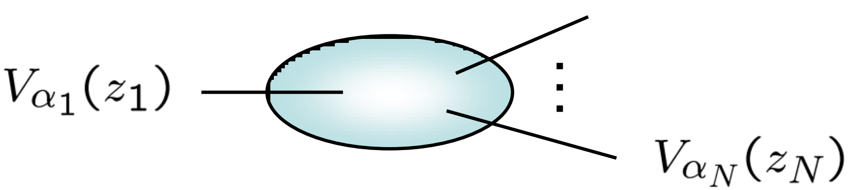
$$S_{\text{free}} = \frac{1}{2\pi}\int d^2w\,\bar{\partial}\varphi\partial\varphi, \;\; S_{\text{int}} = \frac{b^2}{2\pi}\int d^2w\sqrt{g}e^{2b\varphi}$$

$$S_{\text{LD}} = \int d^2w\frac{\sqrt{g}}{8\pi}\mathcal{R}Q_\varphi\varphi, \;\; Q_\varphi = b + \frac{1}{b}$$



- N-pt. scattering amplitudes

$$\left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu) \right\rangle =$$

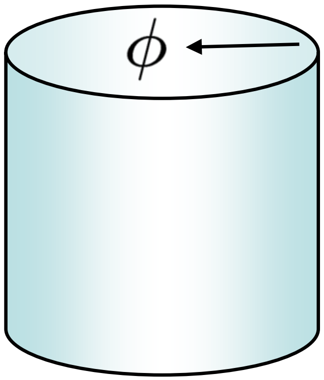


$$V_\alpha(z) := e^{2\alpha\varphi}, \;\; \Delta = \alpha(Q_\varphi - \alpha)$$

H_3^+ model

H_3^+ model (=SL(2,C)/SU(2) WZNW model) describes strings on Euclidean AdS_3

$$ds^2 = (d\phi)^2 + e^{-2\phi} d\gamma d\bar{\gamma}$$



- The action of WZNW model

$$S^H = \frac{k}{4\pi} \int d^2z \langle g^{-1}\partial g, g^{-1}\bar{\partial} g\rangle + \Gamma_{WZ}$$
$$= \frac{k}{2\pi} \int d^2w \left(\bar{\partial}\phi\partial\phi + e^{-2\phi}\bar{\partial}\gamma\partial\bar{\gamma} \right)$$

Introduce $\beta, \bar{\beta}$ $(\beta \propto \partial\bar{\gamma}e^{-2b\phi}, \bar{\beta} \propto \bar{\partial}\gamma e^{-2b\phi})$

$$S^H = \frac{1}{2\pi} \int d^2w \left(\bar{\partial}\phi\partial\phi - \beta\bar{\partial}\gamma - \bar{\beta}\partial\bar{\gamma} - b^2\beta\bar{\beta}e^{2b\phi} \right)$$

Free action with an interaction term $Q_\phi = b = \frac{1}{\sqrt{k-2}}$

- Vertex operator

$$V_j(\mu|z) := |\mu|^{2j+2} e^{\mu\gamma - \bar{\mu}\bar{\gamma}} e^{2b(j+1)\phi}, \quad \Delta = -b^2 j(j+1)$$

H_3^+- Liouville relation

- H_3^+- Liouville relation

$$\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^H = \delta^2\Big(\sum_{\nu=1}^{N} \mu_\nu\Big)|u\Theta_N|^2 \left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu) \prod_{i=1}^{N-2} V_{-\frac{1}{2b}}(y_i) \right\rangle^L$$

$$\Theta_N(y_j, z_\nu) = \prod_{\mu<\nu}(z_\mu - z_\nu)^{\frac{1}{2b^2}} \prod_{i<j}(y_i - y_j)^{\frac{1}{2b^2}} \prod_{\mu,i}(z_\mu - y_i)^{-\frac{1}{2b^2}}$$

- N-2 degenerate fields $V_{-1/2b}$ inserted at y_i

$$\sum_{\nu=1}^{N} \frac{\mu_\nu}{w - z_\nu} = u\frac{\prod_{i=1}^{N-2}(w - y_i)}{\prod_{\nu=1}^{N}(w - z_\nu)}$$

(Sklyanin's separation of variables)

Transverse momentum μ of AdS_3 is mapped to the inserted point y.

- The shifts of parameters

$$b(j_\nu + 1) \ \rightarrow \ \alpha_\nu = b(j_\nu + 1) + \frac{1}{2b}, \quad Q_\phi = b \ \rightarrow \ Q_\varphi = b + \frac{1}{b}$$

Path integral derivation (I)

- Path integral form of *N*-point function

$$\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^{H} = \int \mathcal{D}\phi \mathcal{D}^2\beta \mathcal{D}^2\gamma e^{-S^H} \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu)$$

- Integration over γ, β

Terms including γ: $S^H \sim \gamma\bar{\partial}\beta$, $V \sim \exp(\mu\gamma)$

$\Longrightarrow$ Integration over γ leads to δ-function for β

$$\bar{\partial}\beta = 2\pi \sum_{\nu=1}^{N} \mu_\nu \delta^2(w - z_\nu)$$

$$\left(\bar{\partial}(1/z) = 2\pi\delta^2(z)\right)$$

$\Downarrow$ Integration over world-sheet coordinates

$$\beta = \sum_{\nu=1}^{N} \frac{\mu_\nu}{w - z_\nu} = u \frac{\prod_{i=1}^{N-2}(w - y_i)}{\prod_{\nu=1}^{N}(w - z_\nu)} =: u\mathcal{B}(y_i, z_\nu; w)$$

Path integral derivation (II)

After integrating out β, γ, the theory includes only the radial direction ϕ.
However, the theory is not the same as the Liouville field theory yet.

- Field redefinition of ϕ

1) Interaction term: $-\beta\bar{\beta}e^{2b\phi} = |u\mathcal{B}|^2 e^{2b\phi} = e^{2b\varphi}$

$$\varphi := \phi + \tfrac{1}{2b}\ln|u\mathcal{B}|^2$$
$$= \phi + \tfrac{1}{2b}\Big(\sum_{i=1}^{N-2}\ln|w-y_i|^2 - \sum_{\nu=1}^{N}\ln|w-z_\nu|^2 + \ln|u|^2\Big)$$

2) Kinetic term:

$$\Big(\partial\bar{\partial}\ln|z|^2 = 2\pi\delta^2(z)\Big)$$

$$\frac{1}{2\pi}\int d^2w\,\phi\partial\bar{\partial}\phi =$$
$$= \frac{1}{2\pi}\int d^2w\,\varphi\partial\bar{\partial}\varphi - \tfrac{1}{b}\Big(\sum_{i=1}^{N-2}\varphi(y_i) - \sum_{\nu=1}^{N}\varphi(z_\nu)\Big) + \cdots$$

The shift of *j*-momentum

Insertion of degenerated fields

Twist factor Θ_N

Path integral derivation (III)

- Shift of background charge

1) Introduce Weyl factor

$$ds^2 = |\rho(z)|^2 dz d\bar{z}, \ \ \sqrt{g}\mathcal{R} = -4\partial\bar{\partial}\ln|\rho|^2$$

2) Change ϕ such that $-\beta\bar{\beta}e^{2b\phi} = \sqrt{g}e^{2b\varphi}$

$$\varphi := \phi + \frac{1}{2b}\left(\sum_{i=1}^{N-2}\ln|w-y_i|^2 - \sum_{\nu=1}^{N}\ln|w-z_\nu|^2 - \ln|\rho|^2\right)$$

3) Compute the kinetic term

$$\frac{1}{2\pi}\int d^2w\phi\partial\bar{\partial}\phi = \frac{1}{2\pi}\int d^2w\left(\varphi\partial\bar{\partial}\varphi + \frac{1}{4b}\sqrt{g}\mathcal{R}\varphi\right) + \cdots$$

4) Read the shift of background charge

$$Q_\phi = b \ \to \ Q_\varphi = b + \frac{1}{b}$$

Few comments

- H_3^+- Liouville relation

$$\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^H = \delta^2\Big(\sum_{\nu=1}^{N} \mu_\nu\Big)|\Theta_N|^2 \left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu) \prod_{i=1}^{N-2} V_{-\frac{1}{2b}}(y_i) \right\rangle^L$$

$$\alpha_\nu = b(j_\nu + 1) + \frac{1}{2b}, \qquad \sum_{\nu=1}^{N} \frac{\mu_\nu}{w - z_\nu} = u \frac{\prod_{i=1}^{N-2}(w - y_i)}{\prod_{\nu=1}^{N}(w - z_\nu)}$$

  - Relation between differential equations
    - KZ eq. for H_3^+ model ⇔ BPZ eq. for Liouville theory
  - Extension to generic Riemann surface of genus *g*
    - The number of inserted fields is *N*-2+2*g*
  - Supersymmetric generalization
    - OSP(p|2) WZNW model ⇔ *N*=p super Liouville theory
  - Inclusion of spectrally flowed operators [Ribaut]
    - The number of inserted fields is *N*-2-S (total # of flow)

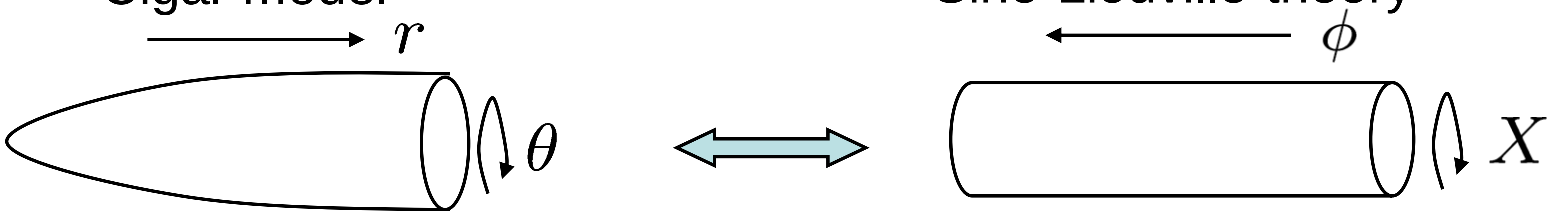
3. The FZZ duality

Duality between the cigar model and Sine-Liouville theory

Outline of proof

- The FZZ duality

Cigar model | Sine-Liouville theory



$$ds^2 = k[dr^2 + \tanh^2 r d\theta^2]$$

$$\Phi = \Phi_0 - 2\log\cosh r$$

$$S_{\text{int}} = 2\lambda \int d^2w e^{\frac{1}{b}\phi} \cos(\sqrt{k}\tilde{X})$$

$$Q_\phi = b = 1/\sqrt{k-2}$$

- Strategy

  1) The cigar model ⇨ H_3^+ model + free boson

  2) H_3^+- Liouville relation ⇨ Relation between the cigar model and a new theory [Liouville + free boson]

  3) The new theory ⇨ Sine-Liouville theory

The cigar model

- The gauged WZNW model
  - H_3^+ WZNW model + Free boson

$$S^{\text{cig}} = S^{\text{WZNW}}[\phi,\beta,\gamma] + \frac{1}{2\pi}\int d^2w\,\partial X\bar{\partial}X \;\; (+\ \text{FP ghosts})$$

  - Primary fields
    - Basis change: μ-basis $\longrightarrow$ m-basis

$$\Phi^j_{m,\bar{m}} = N^j_{m,\bar{m}}\int\frac{d\mu^2}{|\mu|^2}\mu^m\bar{\mu}^{\bar{m}}V_j(\mu|z),\;\; N^j_{m,\bar{m}} = \frac{\Gamma(-j-m)}{\Gamma(j+1+\bar{m})}$$

    - Gauge invariant operators

$$\Psi^j_{m,\bar{m}} = V^X_{m,\bar{m}}\Phi^j_{m,\bar{m}},\;\; V^X_{m,\bar{m}} = e^{i\frac{2}{\sqrt{k}}(mX_L-\bar{m}X_R)}$$

  - Correlation functions

$$\left\langle\prod_{\nu=1}^N\Psi^{j_\nu}_{m_\nu,\bar{m}_\nu}(z_\nu)\right\rangle^{\text{cig}} = \prod_{\nu=1}^N\left[N^{j_\nu}_{m_\nu,\bar{m}_\nu}\int\frac{d^2\mu_\nu}{|\mu_\nu|^2}\mu_\nu^{m_\nu}\bar{\mu}_\nu^{\bar{m}_\nu}\right]\left\langle\prod_{\nu=1}^N V^X_{m_\nu,\bar{m}_\nu}(z_\nu)V_{j_\nu}(\mu_\nu|z_\nu)\right\rangle^{H\times F}$$

Correlation functions of cigar model

- Map to Liouville + Free boson theories
  - Correlators of cigar model

    = Correlators of Free boson (X) x H_3^+ model (ϕ, γ, β)

    $\Downarrow$ H_3^+- Liouville relation

    = Correlators of Free boson (X) x Liouville theory (φ)

  - Redefinition from X to χ

$$\chi_L(w) := X_L(w) - i\frac{\sqrt{k}}{2}\left(\sum_i \ln(w - y_i) - \sum_\nu \ln(w - z_\nu) + \ln u\right)$$

  - Jacobian for change of variables from μ to y

$$\prod_{\nu=1}^{N} \frac{d^2\mu_\nu}{|\mu_\nu|^2}\delta^2\left(\sum_\nu \mu_\nu\right) = |\Theta_N(y_j, z_\nu)|^{4b^2}\frac{d^2u}{|u|^4}\prod_{i=1}^{N-2} d^2y_i$$

The cigar – Liouville correspondence

- The correspondence

$$\left\langle \prod_{\nu=1}^{N} \Psi^{j_\nu}_{m_\nu,\bar{m}_\nu}(z_\nu) \right\rangle^{\text{cig}} = \int \prod_{i=1}^{N-2} \frac{d^2 y_i}{(N-2)!} \prod_{\nu=1}^{N} N^{j_\nu}_{m_\nu,\bar{m}_\nu} \times$$
$$\times \left\langle \prod_{\nu=1}^{N} V^{\chi}_{m_\nu - \frac{k}{2}, \bar{m}_\nu - \frac{k}{2}}(z_\nu) V_{\alpha_\nu}(z_\nu) \prod_{i=1}^{N-2} V^{\chi}_{\frac{k}{2},\frac{k}{2}}(y_i) V_{-\frac{1}{2b}}(y_i) \right\rangle$$

  - *N*-pt. functions of the cigar model

$\Updownarrow$

  - (2*N*-2)-pt. function of Liouville + free boson

$$S = \int \frac{d^2 w}{2\pi} \left(\partial\varphi\bar{\partial}\varphi + \partial\chi\bar{\partial}\chi + \frac{\sqrt{g}}{4}\mathcal{R}(Q_\varphi \varphi + Q_{\tilde{\chi}}\tilde{\chi}) + \mu e^{2b\varphi} \right)$$
$$Q_\varphi = b + 1/b, \;\; Q_{\tilde{\chi}} = -i\sqrt{k}$$

Three shortfalls

- Three more steps to derive FZZ duality

1) The relation is not weak-strong duality for *k*

⟹ We utilize the self-duality of Liouville theory

$$\mu e^{2b\varphi} \leftrightarrow \tilde{\mu} e^{\frac{2}{b}\varphi}, \quad Q_\varphi = b + 1/b$$

2) It relates *N*-pt. functions to (2*N*-2)-pt. functions

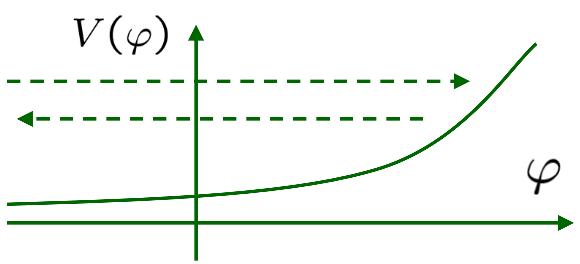
⟹ Insertions can be interpreted as an interaction

$$V_{\text{int}} = \tilde{\mu} e^{\frac{2}{b}\varphi} - 2e^{-\frac{1}{b}\varphi + i\sqrt{k}\tilde{\chi}}$$

3) The dual theory is not Sine-Liouville theory

⟹ Field redefinitions + reflection relation

$$\begin{cases} \phi = (k-1)\varphi - i\sqrt{k}b^{-1}\tilde{\chi} \\ \tilde{X} = -i\sqrt{k}b^{-1}\varphi - (k-1)\tilde{\chi} \end{cases}$$



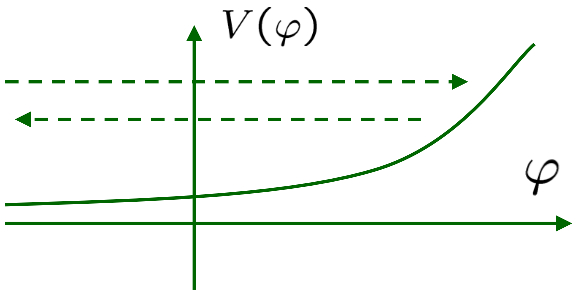
Reflection relation

- Field identification in Liouville theory

$$S_L = \frac{1}{2\pi}\int dw^2 \left(\partial\varphi\bar{\partial}\varphi + \frac{\sqrt{g}}{4}\mathcal{R}Q_\varphi\varphi + \mu e^{2b\varphi}\right),\ \ Q_\varphi = b + 1/b$$

  – Reflection relation $w : \alpha \mapsto Q - \alpha$

$$V_\alpha = D(\alpha)V_{Q_\varphi-\alpha},\ \ V_\alpha = e^{2\alpha\varphi}$$



- A generic model

$$S = \frac{1}{2\pi}\int d^2\left(\sum_{i=1}^{n}\partial X_i\bar{\partial}X_i + \sum_{\nu=1}^{p}\mu_\nu e^{2(\vec{\beta}_\nu,\vec{X})}\right),\ \ \vec{\beta}_\nu(\vec{Q}-\vec{\beta}_\nu) = 1$$

  – An interaction term $e^{2(\vec{\beta}_\nu,\vec{X})}$ can be reflected by $e^{2(\vec{\beta}_\rho,\vec{X})}$

$$w_\rho : \vec{\beta}_\nu \longrightarrow \vec{\beta}_\nu + \vec{\beta}_\rho + (1 - 2(\vec{\beta}_\nu,\vec{\beta}_\rho))\frac{\vec{\beta}_\rho}{(\vec{\beta}_\rho,\vec{\beta}_\rho)}$$

The FZZ duality

- The duality between the cigar and Sine-Liouville
  - Relation between correlators of primary fields

$$\left\langle \prod_{\nu=1}^{N} \Psi^{\mathrm{cig};j_\nu}_{m_\nu,\bar{m}_\nu} \right\rangle^{\mathrm{cig}} = \mathcal{N} \left\langle \prod_{\nu=1}^{N} e^{2b(j_\nu+1)\phi} e^{i\frac{2}{\sqrt{k}}(m_\nu X_L - \bar{m}_\nu X_R)} \right\rangle^{\mathrm{SL}}$$

$$S = \int \frac{d^2w}{2\pi}\left(\partial\phi\bar{\partial}\phi + \partial X\bar{\partial}X + \frac{\sqrt{g}}{4}\mathcal{R}Q_\phi\phi + 4\pi\lambda e^{\frac{1}{b}\phi}\cos(\sqrt{k}\tilde{X})\right)$$

  - Duality between the theories
    - The both theories have the same $\hat{W}_\infty(k)$ symmetry

[Fateev-Lukyanov]

  - Violation of winding number conservation

$$\sum_\nu m_\nu = \sum_\nu \bar{m}_\nu = S,\ \ |S| \leq N-2$$

    - We need to include spectral flow action to H_3^+ model.
      (#(inserted degenerated fields) = *N*-2-|*S*|)

4. Conclusion

Summary and future problems

Conclusion

- Summary
  - H_3^+- Liouville relation
    - Path integral derivation
    - Generalizations and applications
      (Higher genus, supersymmetric model, …)
  - FZZ duality
    - Duality between the cigar model and Sine-Liouville theory
    - Condensation of winding string tachyon
- Future problems
  - Generalizations $(J_1^- = \mu_1,\ J_2^- = \mu_2,\ J_3^- = \mu_3)$
    - SL(N) WZNW model from SL(N) Toda theory
    - Other SUSY models and their applications
    - Generalizations of FZZ duality [Fateev]
  - Applications
    - Matrix model, AdS/CFT correspondence
    - Implication to geometric Langlands duality [Giribet-Nakayama]

5. Appendix

Technical details

Relation between differential equations (I)

- H_3^+ model
  - Knizhnik-Zamolodchikov equations

$$T(z) - b^2 :J^\mu J_\mu:(z) = 0 \qquad (z = z_\nu, y_j)$$

$$\Leftrightarrow D_z^H \Omega^H(z_\nu, \mu_\nu) = 0,\ \Omega^H(z_\nu, \mu_\nu) = \left\langle \prod_\nu V_{j_\nu}(\mu_\nu | z_\nu) \right\rangle$$

Sklyanin's separation of variables +

$$\Omega^H(z_\nu, \mu_\nu) = |\Theta_N(z_\nu, y_j)|^2 \Omega^L(z_\nu, y_j)$$

- Liouville field theory
  - Belavin-Polyakov-Zamolodchikov equations

$$(b^2 \partial_{y_j}^2 + T(y_j)) V_{-\frac{1}{2b}}(y_j) = 0$$

$$\Leftrightarrow D_j^L \Omega^L(z_\nu, y_j) = 0,\ \Omega^L(z_\nu, y_j) = \left\langle \prod_\nu V_{\alpha_\nu}(z_\nu) \prod_j V_{-\frac{1}{2b}}(y_j) \right\rangle$$

Relation between differential equations (II)

- A sketch of proof
  - Sklyanin's separation of variables

$$\sum_{\nu=1}^{N} \frac{\mu_\nu}{w - z_\nu} = u \frac{\prod_{j=1}^{N-2}(w - y_j)}{\prod_{\nu=1}^{N}(w - z_\nu)}$$

$$\begin{pmatrix} e^- = \mu, \\ e^0 = -\mu\partial_\mu, \\ e^+ = \mu\partial_\mu^2 - j(j+1)/\mu \end{pmatrix}$$

  - Sugawara singular vector at $z=y_j$

$$T(y_j) + b^2\left[J^0(y_j)J^0(y_j) - \tfrac{1}{2}(J^-(y_j)J^+(y_j) + J^+(y_j)J^-(y_j))\right] = 0$$

$$\begin{cases} J^-(y_j) = \sum_{\nu=1}^{N} \frac{\mu_\nu}{y_j - z_\nu} = 0, \\ J^0(y_j) = -\sum_{\nu=1}^{N} \frac{\mu_\nu \partial_{\mu_\nu}}{y_j - z_\nu} = -\partial_{y_j} \end{cases}$$

  - Change of variables + twist factor leads to BPZ equation

$$b^2\partial_{y_j}^2 + T(y_j) = 0 \longleftarrow \text{Separation of variables}$$

Extension to higher genus g (I)

- H_3^+-Liouville relation for $g \geq 1$

$$\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) \right\rangle^H = |\Theta_N^g|^2 \left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu) \prod_{j=1}^{N-2+2g} V_{-\frac{1}{2b}}(y_j) \right\rangle^L$$

⇨ The number of extra insertion is N-2+2g

- Meromorphic 1-form on surface with g

#(pole) - #(zero) = 2-2g

– For g=0

#(zero)=N-2+2g

$$\beta = \sum_{\nu=1}^{N} \frac{\mu_\nu}{w - z_\nu} = u \frac{\prod_{j=1}^{N-2}(w - y_j)}{\prod_{\nu=1}^{N}(w - z_\nu)}$$

#(pole)=N

Extension to higher genus g (II)

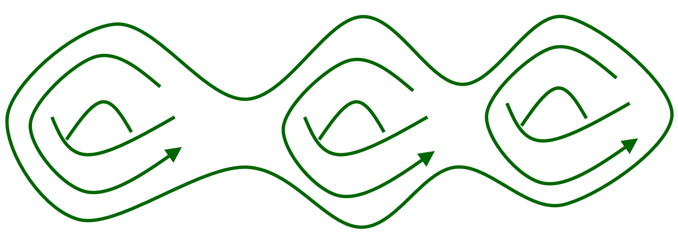
- The map of parameters
  - For $g = 0$
    - μ (momentum): N
    - μ-conservation: -1
    - N-1

    Sklyanin's separation of variables

    - u: 1
    - y_j (pos. of deg. field): N-2
    - N-1
  - For $g \geq 1$
    - μ (momentum): N
    - λ (holonomy): g
    - β_0 (zero mode): g-1
    - N-1+2g

    Sklyanin's separation of variables

    - u: 1
    - y_j (pos. of deg. field): N-2+2g
    - N-1+2g



$$\begin{cases} \beta(w+\tau) = e^{2\pi i\lambda}\beta(w), \\ \gamma(w+\tau) = e^{-2\pi i\lambda}\gamma(w), \\ \psi(w+\tau) = \phi(w) + \frac{2\pi}{b}\mathrm{Im}\lambda \end{cases}$$

OSP(1|2) – *N*=1 Liouville relation (I)

- A suprsymmetric generalization

SL(2,C)/SU(2) WZNW model (H_3^+ model)

$\Downarrow$ Target-space supersymmetry

OSP(1|2) WZNW model: $\underbrace{J^{\pm}, J^{3}}_{\text{SL(2) generators}}, F^{\pm}$ ← Fermionic generators

- OSP(1|2) model from *N*=1 Liouville theory

OSP(1|2) model		*N*=1 Liouville theory
$\phi, \gamma, \bar{\gamma}, \theta, \bar{\theta}$	$\Rightarrow$	$\varphi, \psi, \bar{\psi}$
$(+\beta, \bar{\beta}, p, \bar{p})$		+ free fermion $\chi, \bar{\chi}$

$$\begin{cases} \text{1. Integrate over } \gamma, \bar{\gamma}, \beta, \bar{\beta} \\ \text{2. Redefine: } (\phi, \theta, \bar{\theta}, p, \bar{p}) \longmapsto (\varphi, \psi, \bar{\psi}, \chi, \bar{\chi}) \end{cases}$$

OSP(1|2) – N=1 Liouville relation (II)

- OSP(1|2) – N=1 Liouville relation

N-pt. function of OSP(1|2) WZNW model

$\Updownarrow$

(2N-2)-pt. function of N=1 Liouville
+ free fermion with (N-2) degenerate fields

- Comments
  - 2 and 3-point functions of OSP(1|2) model are computed and consistent with OSP(1|2) symmetry.
    - The relevant results of N=1 Liouville are given in [Rashkov-Stanishkov, Poghosian, Fukuda-Hosomichi]
  - This is the first practical use of H_3^+ - Liouville relation.

Winding number violation (I)

- SL(2) current algebra

$$[J^+_m, J^-_n] = km\delta_{m+n} - 2J^3_{m+n},$$
$$[J^3_m, J\pm_n] = \pm J^\pm_{m+n},\ [J^3_m, J^3_n] = -\frac{k}{2}\delta_{m+n}$$

Vacuum state: $J^a_n|0\rangle = 0 \ \text{for} \ n \geq 0$

- Spectral flow symmetry

$$\rho^S(J^3_n) = J^3_n - \tfrac{k}{2}S\delta_{n,0},\ \rho^S(J^\pm_n) = J^\pm_{n\pm S}$$

Vacuum state: $\rho^S(J^a_n)|S\rangle = 0 \ \text{for} \ n \geq 0$

- Free field realization

$$J^- = \beta,\ J^3 = \beta\gamma - b^{-1}\partial\phi, \quad J^+ = \beta\gamma^2 - 2b^{-1}\gamma\partial\phi + k\partial\gamma$$

$$\Longrightarrow \beta_{n-S}|S\rangle_{\gamma,\beta} = \gamma_{n+S}|S\rangle_{\gamma,\beta} = 0 \ (n \geq 0),\ |S\rangle_\phi = e^{\frac{S}{b}\phi}|0\rangle_\phi$$

β(*w*) must have a zero of order *S* at *w*=0

Winding number violation (II)

- Ribault relation

$$\left\langle \prod_{\nu=1}^{N} V_{j_\nu}(\mu_\nu|z_\nu) v^S(0) \right\rangle^H = \prod_{n=0}^{S} \delta^2\left(\sum_{\nu=1}^{N} \mu_\nu z_\nu^{-n}\right) \times$$

$$\times \frac{|\Theta_N|^2}{|u|^{\frac{S}{b^2}-2}} \left\langle \prod_{\nu=1}^{N} V_{\alpha_\nu}(z_\nu) \prod_{i=1}^{N-2-S} V_{-\frac{1}{2b}}(y_i) \right\rangle^L$$

$$|S\rangle \equiv v^S(0)|0\rangle \qquad \sum_{\nu=1}^{N} \frac{\mu_\nu}{w-z_\nu} = u \frac{w^S \prod_{i=1}^{N-2-S}(w-y_i)}{\prod_{i=1}^{N}(w-z_\nu)}, \quad S \le N-2$$

- Winding number violation of the cigar model

$$\sum_\nu m_\nu = \sum_\nu \bar{m}_\nu = S, \quad |S| \le N-2$$

  – Insert a representation of the identity of the cigar model

$$1 = V^X_{-\frac{kS}{2},-\frac{kS}{2}}(0) v^S(0) \Longrightarrow$$ The FZZ duality with non-zero *S*

Winding # violation

Ribault relation

The gauged WZNW model

- The cigar model as gauged WZNW model
  - The action of H_3^+ model

$$S^{\text{WZNW}}[g] = \frac{k}{2\pi}\int d^2w \left(\partial\phi\bar{\partial}\phi + e^{-2\phi}\bar{\partial}\gamma\partial\bar{\gamma}\right)$$

$$\Big\downarrow \quad \text{Gauging U(1) isometry: } \mathcal{A} = A dw + \bar{A} d\bar{w}$$

$$S^{\text{cig}}[g,\mathcal{A}] = \frac{k}{2\pi}\int d^2w \left((\partial\phi + A)(\bar{\partial}\phi + \bar{A}) + e^{-2\phi}(\bar{\partial} + \bar{A})\gamma(\partial + A)\bar{\gamma}\right)$$

  - Field redefinition

$$A = \partial x_L, \quad \bar{A} = \bar{\partial} x_R \quad \text{(+ FP ghosts)}$$

Shift the fields

$$\phi \mapsto \phi - \tfrac{1}{2}(x_L + x_R), \quad \gamma \mapsto \gamma e^{-x_R}, \quad \bar{\gamma} \mapsto \bar{\gamma} e^{-x_L}$$

$$S^{\text{cig}}[g,X] = S^{\text{WZNW}}[g] + \tfrac{1}{2\pi}\int d^2w \partial X \bar{\partial} X \ \text{(+ FP ghosts)}$$

$$X = \tfrac{\sqrt{k}}{2i}(x_L - x_R)$$

Relation to Sine-Liouville theory

- Field redefinition

$$\begin{cases} \phi = (k-1)\varphi - i\sqrt{k}b^{-1}\tilde{\chi} \\ \tilde{X} = -i\sqrt{k}b^{-1}\varphi - (k-1)\tilde{\chi} \end{cases} \Longrightarrow \begin{cases} V_L = e^{\frac{2}{b}\varphi} = e^{\frac{2(k-1)}{b}\phi - \frac{2i\sqrt{k}}{b^2}\tilde{X}} \\ V_- = e^{-\frac{1}{b}\varphi + i\sqrt{k}\tilde{X}} = e^{\frac{1}{b}\phi - i\sqrt{k}\tilde{X}} \end{cases}$$

$$(Q_\phi = b,\ Q_X = 0)$$

- Reflection by V_- is performed for V_L

$$V_L = e^{\frac{2(k-1)}{b}\phi - \frac{2i\sqrt{k}}{b^2}\tilde{X}} \propto e^{\frac{1}{b}\phi + i\sqrt{k}\tilde{X}} \equiv V_+$$

$$\Longrightarrow V_+ + V_- = 2e^{\frac{1}{b}\phi}\cos(\sqrt{k}\tilde{X})$$

Interaction term for Sine-Liouville

- Reflection by V_- is performed also for N vertex ops.

$$N^j_{m,\bar{m}} V_\alpha V^\chi_{m-\frac{k}{2},\bar{m}-\frac{k}{2}} = -e^{2b(j+1)\phi + i\frac{2}{\sqrt{k}}(mX_L - \bar{m}X_R)}$$

Field redefinition + reflection