Quantum Crystals and Topological Strings

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Outline

Outline

Welcome

Random partitions

- Integer and plane partitions.
- Mapping to crystal melting.
- The partition function is the same as for the A-model in topological strings.
- We quantize the statistical system.
- Read the corresponding quantum gravity.

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Results

Quantum crystal melting

- Quantize the crystal melting system.
- The two dimensional problem is equivalent to XXZ. Integrable.
- The three dimensional problem can be written in terms of interacting spin chains.
- Numerical analysis.
- Stochastic quantization.
- Effective model for quantum gravity.
- Much to be understood.

Outline

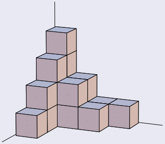
1. Why are we here?
2. **Classical Crystal melting and Quantization**
3. Two dimensions and XXZ
4. Three dimensions
5. Stochastic quantization
6. Conclusions

Classical crystals

Crystal growth

- The crystal is represented by cubes on an integer lattice in the positive octant

- A cube can be added if it is attached on three faces to the wall or other cubes



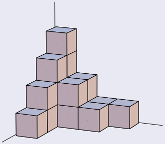
- Partition function [MacMahon]

$$Z = \sum_{\text{configurations}} q^{\text{number of boxes}} = \sum_n \frac{1}{(1-q^n)^n} .$$

Classical crystals

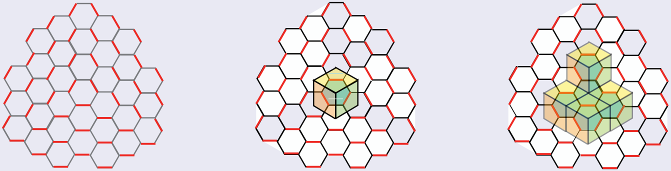
Mapping to plane partitions and dimer model

- MacMahon found the result by studying random plane partitions



4	2	1	1
3	2	1	0
2	0	0	0
1	0	0	0

- Mapping to dimers on hexagonal lattice (height function)

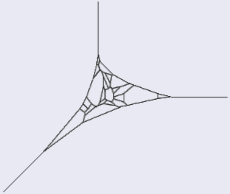


Melting Crystal and Quantum Calabi-Yau

Quantum foam

- The same partition function is found to be the one of topological A-model on $\mathbb{C}^3$ [Iqbal, Nekrasov, Okounkov, Reshetikhin, Vafa]
- A-model path integrals localize on "quantum Kähler structures" on $\mathbb{C}^3$.
- In terms of toric geometry think of quantized multiple blow-ups of the origin

There is a correspondence with three dimensional partitions



Quantization

Classical probabilities

- Classical system. Configurations $\{ C_i \}$ with energy $\bar{\mathcal{H}}$.
- For the canonical ensemble, the probability of C_i is

$$p(C_i) = \frac{e^{-\beta \bar{\mathcal{H}}(C_i)}}{Z}, \qquad Z = \sum_i e^{-\beta \bar{\mathcal{H}}(C_i)} .$$

Quantum probabilities

- Quantum system. $|C_i\rangle$ are orthonormal generators of $\mathscr{H}$.
- A vector $|\Psi\rangle$ in $\mathcal{H}$ defines a probability measure on the $|C_i\rangle$

$$m_\psi(C_i) = \frac{|\langle \psi | C_i \rangle|^2}{\langle \psi | \psi \rangle}, \qquad \langle \psi | \psi \rangle = \sum_i |\langle \psi | C_i \rangle|^2$$

Quantum Hamiltonian

Ground state

- Introduce $|\Psi_0\rangle$ as

$$|\Psi_0\rangle = \sum_i e^{-\beta\bar{\mathcal{H}}(C_i)/2} |C_i\rangle ,$$

- The probability measure coincides with the one of the canonical ensemble.

Quantum Hamiltonian

- $\mathcal{H}$ be a Hamiltonian operator in the Hilbert space $\mathscr{H}$ s.t.

$$\mathcal{H} |\Psi_0\rangle = 0 .$$

- We call $\mathcal{H}$ the quantum Hamiltonian corresponding to $\bar{\mathcal{H}}$.

Quantum Hamiltonian

Classical limit

- Classical quantity $Q(C_i)$
- Quantum expectation value for an operator $\hat{Q}$

$$\langle \hat{Q} \rangle = \frac{1}{\mathscr{Z}} \operatorname{Tr}_{\mathscr{H}} \left(\hat{Q} \, e^{-\mathcal{H}/\hbar} \right) .$$

- in the $\hbar \to 0$ limit use saddle point to obtain the classical average value

$$\langle \hat{Q} \rangle = \operatorname{Tr}_{\mathscr{H}} \left(\hat{Q} \, e^{-\mathcal{H}/\hbar} \right) \xrightarrow[\hbar \to 0]{} \frac{\langle \Psi_0 | \hat{Q} | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} = \frac{1}{Z} \sum_i e^{-\beta \bar{\mathcal{H}}(C_i)} Q(C_i)$$

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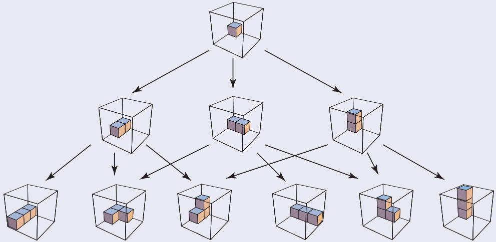
Quantum Hamiltonian

Dynamics

- Consider the state graph



- The quantum Hamiltonian $\mathcal{H}$ is the Laplacian on the graph, plus a potential term counting the boxes

$$\mathcal{H} = -J \left(\triangle + V_q(u) \right)$$

Quantum Hamiltonian

Block Notation

$$\mathcal{H} = -J \sum \left(|\square\rangle\langle\blacksquare| + |\blacksquare\rangle\langle\square| \right) + V \left(\sqrt{q}\, |\square\rangle\langle\square| + \frac{1}{\sqrt{q}}\, |\blacksquare\rangle\langle\blacksquare| \right) ,$$

- $|\square\rangle$ is a place where a cube can be added, $|\blacksquare\rangle$ is a cube that can be removed
- $|\square\rangle\langle\blacksquare| + |\blacksquare\rangle\langle\square|$ is the kinetic term (changes the configuration).
- $|\square\rangle\langle\square|$ and $|\blacksquare\rangle\langle\blacksquare|$ are potential terms (count)
- This is local information: not the total number of boxes but border effects.

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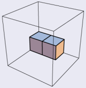
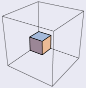
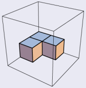
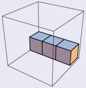
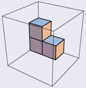
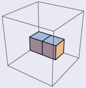
Quantum Hamiltonian

Block Notation

$$\mathcal{H} = -J \sum \left(|\square\rangle\langle\blacksquare| + |\blacksquare\rangle\langle\square| \right) + V \left(\sqrt{q}\, |\square\rangle\langle\square| + \frac{1}{\sqrt{q}}\, |\blacksquare\rangle\langle\blacksquare| \right) ,$$

$\mathcal{H}$  $= -J \Big($  $+$  $+$  $+$  $\Big)$

$+V \left(3\sqrt{q} + \frac{1}{\sqrt{q}} \right)$ 

- Using detailed balance

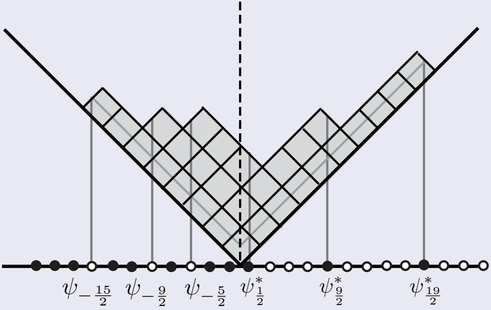
$$\mathcal{H} \left[\sum_{\alpha} q^{N(\alpha)/2} |\alpha\rangle \right] = 0$$

Outline

Two dimensional problem

Partitions and fermions

- Start with a lower dimensional analogue: integer partitions
- Represented by Young diagrams
- Mapping to fermionic operators



2d and XXZ

Hamiltonian for fermions

- Write the Hamiltonian
- Adding a cube is the same as moving a fermion to the right
- Removing a cube is the same as moving a fermion to the left
- The diagonal term counts the possible hops
- The Hamiltonian becomes

$$\begin{aligned}\mathcal{H} = -J \sum_n \psi^*_{n+1}\psi_n + \psi^*_n \psi_{n+1} + \\ - q^{1/2} n_n \left(1 - n_{n+1}\right) - q^{-1/2} n_{n+1} \left(1 - n_n\right)\end{aligned}$$

- This is the XXZ model

Kink ground state

XXZ in the kink sector

- Adding the appropriate boundary conditions we are in the half-full sector with kink b.c.
- The ground state can be expressed in the grand-canonical formalism as

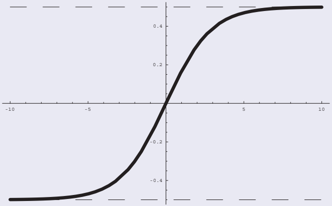
$$\Psi(z) = \prod_{n=0}^{\infty} \left(1 + z\sqrt{q}^{n+1/2} \psi_n^*\right) |0\rangle = \bigotimes_{n=0}^{\infty} \begin{pmatrix} 1 \\ z \sqrt{q}^{n+1/2} \end{pmatrix},$$

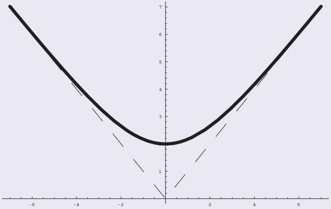
- This expression is particularly nice because the points are essentially decoupled

Kink ground state

Correlation functions

- One can easily compute the correlation functions
- From the one point function one obtains the limit shape (Wulff crystal)





- The n-point functions factorize

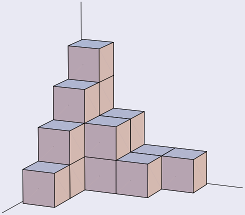
$$\langle \sigma^3_{x_1} \sigma^3_{x_2} \dots \sigma^3_{x_n} \rangle = \langle \sigma^3_{x_1} \rangle \langle \sigma^3_{x_2} \rangle \dots \langle \sigma^3_{x_n} \rangle$$

Outline

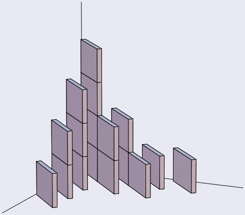
Three dimensions and fermions

Diagonal slicing

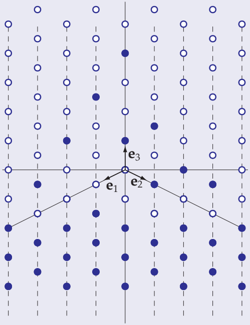
- Let us move to the actual three dimensional problem
- Diagonal slicing



(a) Cubes



(b) Diagonal slicing



(c) Lattice fermions

Three-dimensional Hamiltonian

The fermionic Hamiltonian

- The Hamiltonian describes the vertical hopping of the fermions
- Interlacing conditions: always a plane partition
- Introducing 3 vectors in the plane $\mathbf{e}_1 = \left(-\cos\frac{\pi}{6}, -\sin\frac{\pi}{6}\right)$, $\mathbf{e}_2 = \left(\cos\frac{\pi}{6}, -\sin\frac{\pi}{6}\right)$ and $\mathbf{e}_3 = (0, 1)$.
- The Hamiltonian takes a nice form

$$\mathcal{H}_{3\mathrm{d}} = -J \sum_{\mathbf{x}} \mathcal{H}_{2\mathrm{d}}^{\mathbf{x} \to \mathbf{x}+\mathbf{e}_3} \left(1 - n_{\mathbf{x}-\mathbf{e}_1}\right) \left(1 - n_{\mathbf{x}-\mathbf{e}_2}\right) .$$

- Each particle can hop up or down after checking if on the neighboring lines there is a hole.

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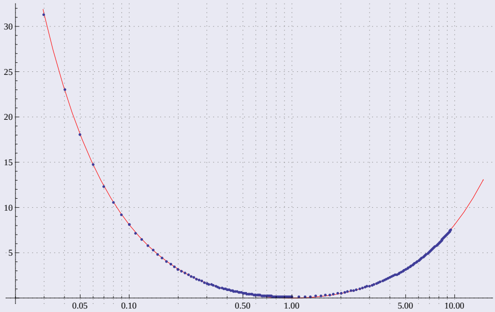
Three dimensions

Excited levels

- The 3d system is definitely harder than the 2d
- The ground state can be written explicitly, but the correlation functions are not trivial
- The excited levels can be studied numerically



Outline

A continuum limit

A (slightly) different story

- Consider a classical system described by an action $S_{\text{cl}}[\phi]$ where $\phi = \phi(x)$.
- A possible way of quantizing it consists in using the so-called stochastic quantization
  - Add an extra (fictitious) euclidean time t
  - Write a Langevin equation that for $t \to \infty$ relaxes to the critical points action

$$\frac{d}{dt}\phi(x,t) = -\frac{\delta S_{\text{cl}}}{\delta \phi} + \eta(x,t)$$

  where η is a white Gaussian noise

$$\langle \eta(x,t) \rangle = 0 \qquad \langle \eta(x,t)\eta(y,t') \rangle = 2\delta(x-y)\delta(t-t')$$

A continuum limit

Correlation functions

- Every correlation functions can be written as an average on the noise:

$$\langle F[\phi(x)] \rangle = \int \mathrm{d}[\eta] \, F[\phi(x)] \, e^{\int dx \, dt \, \eta(x,t)^2}$$

- From the partition function one can extract a (quantum) action in $d+1$ dimension that takes the form

$$S_{\text{quantum}} = \dot{\phi}^2 + \frac{1}{2} \left(\frac{\delta S}{\delta \phi} \right)^2 - \bar{\psi} \left(\frac{\partial}{\partial t} + \frac{\delta^2 S}{\delta \phi^2} \right) \psi$$

where ψ is the supersymmetric partner of ϕ.

A continuum limit

Stochastic quantization and dimers

- Why is this relevant?
- The quantum crystal model I presented describes a Brownian motion among the minima of a classical action (number of boxes)
- The Langevin equation provides a continuum version of the Brownian motion
- For the quantum crystal we just need to identify a classical action and the stochastic quantization will automatically give us the quantum version

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A continuum limit: quantum dimers

The quantum dimer in the continuum limit

- Consider our problem for $q = 1$: the quantum dimer
- A natural conjecture is that the classical system is described by a free boson in two dimensions:

$$S_{\mathrm{cl}}[\phi] = \int dx \, (\nabla\phi(x))^2$$

- Using the procedure outlined we obtain the action for the three-dimensional quantum system:

$$S_{\mathrm{quantum}}[\phi] = \int dxdt \left[\dot{\phi}^2 + \frac{\kappa^2}{2} \left(\nabla^2 \phi \right)^2 - \bar{\psi} \left(\frac{\partial}{\partial t} + \kappa \nabla^2 \right) \psi \right]$$

- The bosonic part has already appeared in literature

Kähler Gravity

Quantum dimers and A-model

- The partition function for the classical dimer is the same as for the A-model
- In a "semiclassical" limit the A-model is described by Kähler gravity: this is our classical theory
- The stochastic quantization of Kähler gravity naturally leads to a theory in seven dimensions, that in a $x_7 \to \infty$ limit reproduces the topological string
- One can conjecture that in this way we describe an effective action for topological M-theory.

Kähler Gravity

An action for M-theory

- In the $g_s \to 0$ limit the action reads

$$S_{7d} = \|\dot{\gamma}\|^2 + \|d_c d\gamma\|^2 - \langle \bar{\psi}, \left(\frac{\partial}{\partial t} + d_c * d \right) \psi \rangle$$

where

- γ is a 3-form
- d_c is the twisted differential $\partial - \bar{\partial}$
- ψ and $\bar{\psi}$ are fermionic superpartners.
- the scalar product and the norm are intended in terms of the Hodge product in 6 dimensions $\langle f, g \rangle = \int f \wedge *_6 g$

Outline

Results

Quantum crystal melting

- Quantize the crystal melting system.
- The two dimensional problem is equivalent to XXZ. Integrable.
- The three dimensional problem can be written in terms of interacting spin chains.
- Numerical analysis.
- Stochastic quantization.
- Effective model for quantum gravity.
- Much to be understood.

Results

Thank you
for your attention