Web 用

A review of

A_∞-algebras of Lagrangian submanifolds

<u>Reference</u> Fukaya, Oh, Ohta and Ono
Lagrangian intersection Floer theory
– anomaly and obstruction –, (2008)

§ Intro

(M,ω) a symplectic manifold

L a Lagrangian submanifold in M

(Suppose that

M and L are compact,

L is relatively spin.)

Fukaya-Oh-Ohta-Ono

$(C_*(L;\mathbb{Q}),\partial)$ the singular chain complex of L

$$\Big\Downarrow \text{ genus } 0 \text{ bordered stable maps}$$

$(C_*(L;\Lambda),\{m_k\}_{k=0}^{\infty})$ an A_∞-algebra

Joyce - A.

A_∞-algebras of Lagrangian immersions

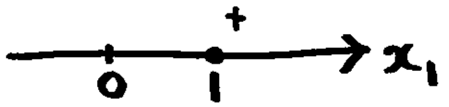
$d < 0$

$$C_d(L;\mathbb{Q}) := \{0\}$$

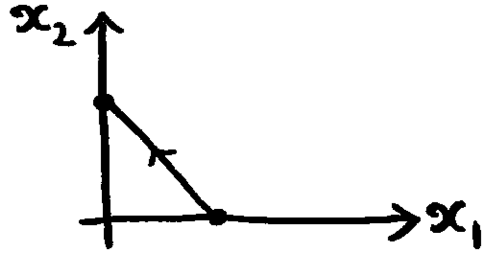
$d \geqq 0$

$$C_d(L;\mathbb{Q}) := \left\{ \sum_i a_i f_i \;\middle|\; a_i \in \mathbb{Q},\ f_i : \Delta_d \to L \right\}$$

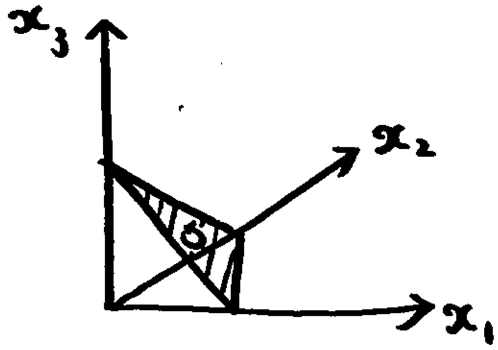
$$\Delta_d := \left\{ (x_1, \dots, x_{d+1}) \in \mathbb{R}^{d+1} \;\middle|\; x_1 + \cdots + x_{d+1} = 1,\ x_i \geqq 0 \right\}$$

$d=0$ 

$\Delta_0 =$ 

$d=1$ 

$\Delta_1 =$ 

$d=2$ 

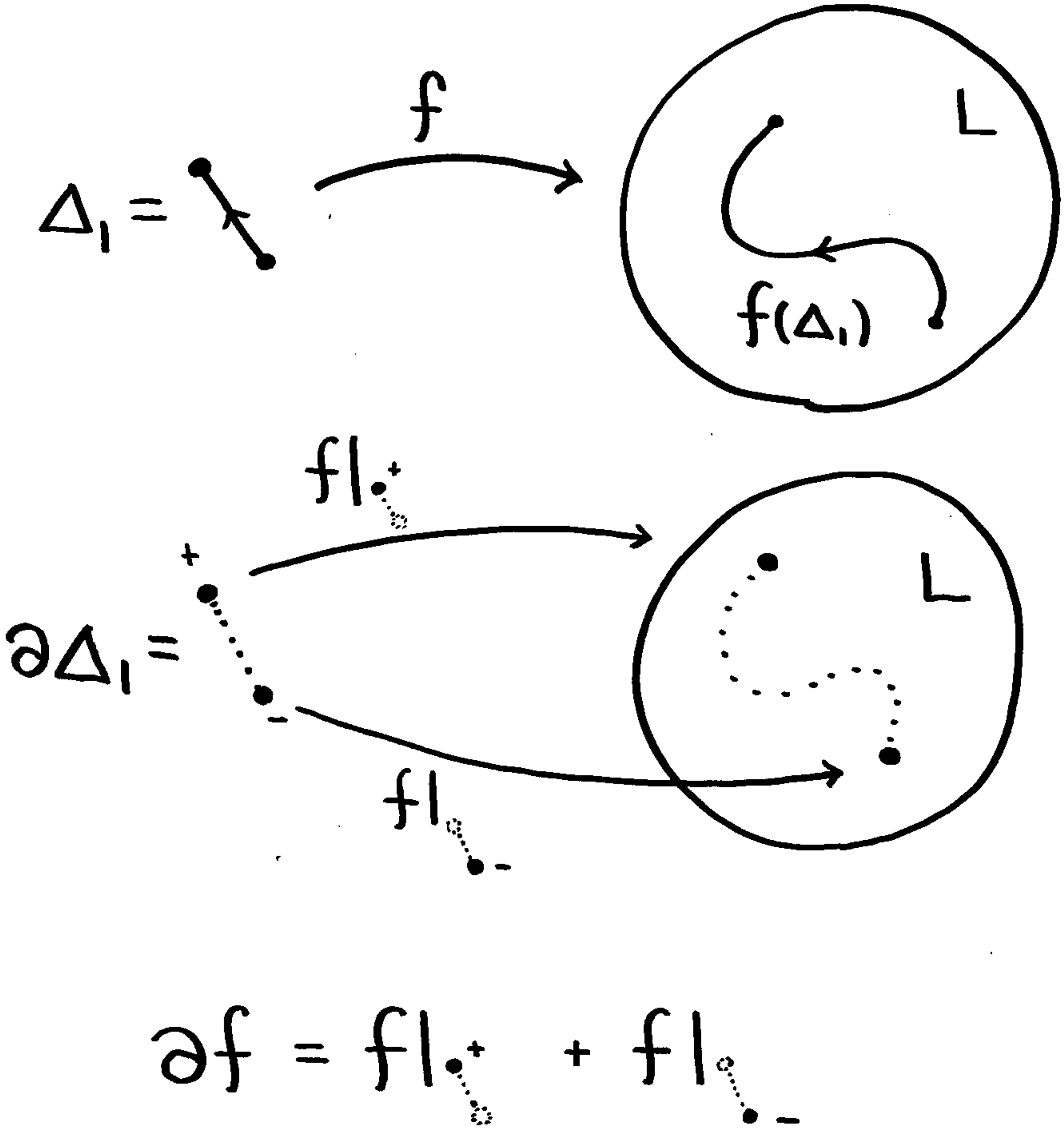
$\Delta_2 =$ 

$\partial : C_d(L;\mathbb{Q}) \to C_{d-1}(L;\mathbb{Q})$ a linear map

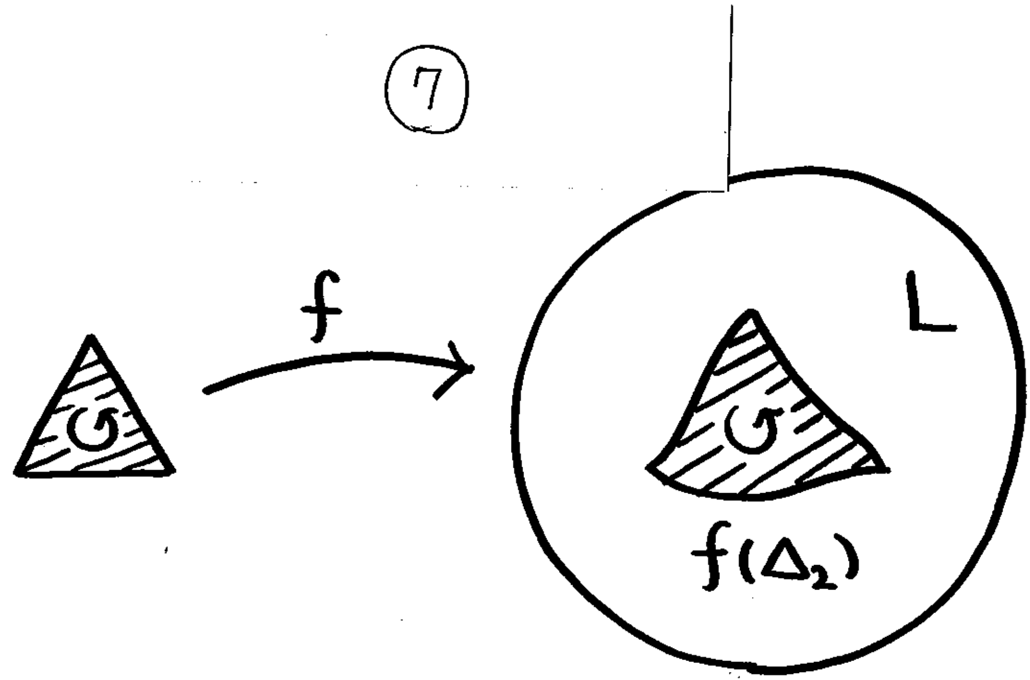
$\partial f :=$ " the restriction

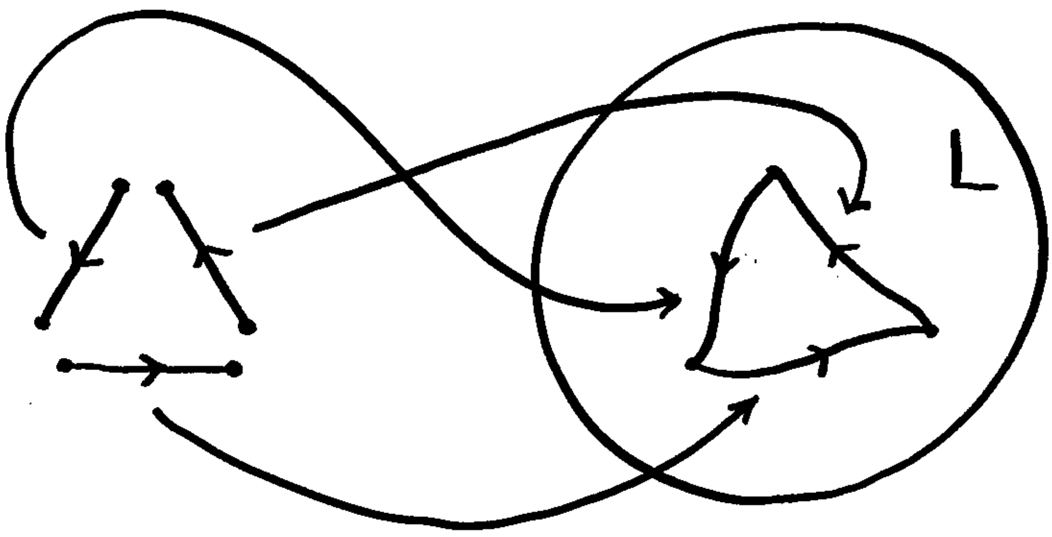
of $f : \Delta_d \to L$

on the faces "



$\Delta_1 =$
f
L
$f(\Delta_1)$
$\partial\Delta_1 =$
$+$
$-$
$f|$
L
$f|$
$\partial f = f| + f|$

$\Delta_2 =$ 

$\partial\Delta_2 =$ 

$$\partial f = f| + f| + f|$$

Remark $\quad \partial \circ \partial = 0$

the singular homology of L

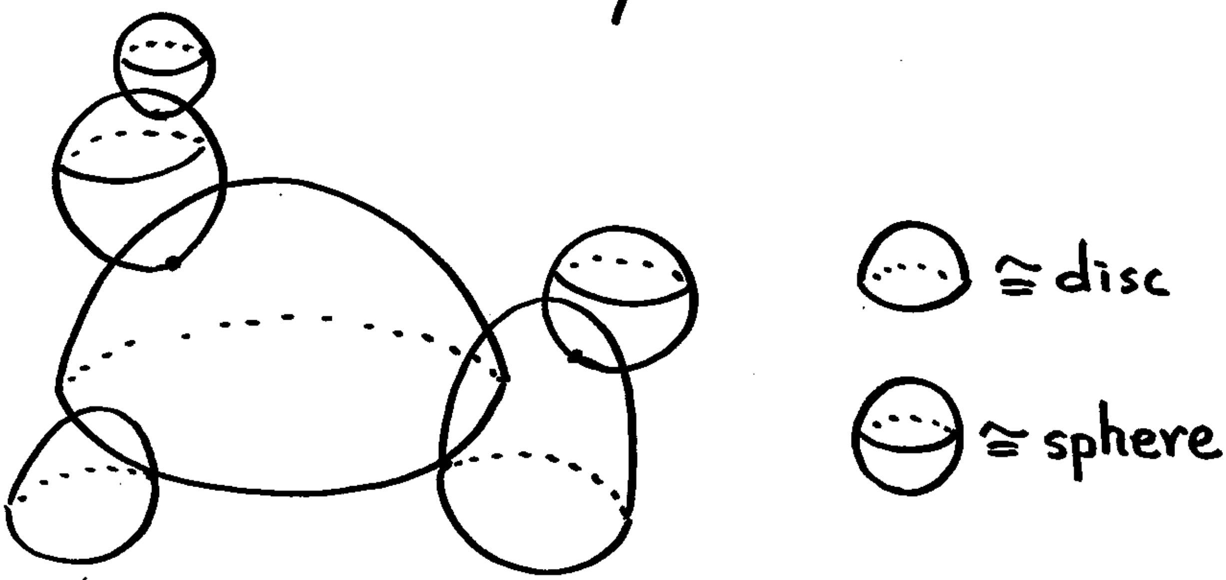
$$H_d(L;\mathbb{Q})$$

$$:= \frac{\operatorname{Ker} \partial : C_d(L;\mathbb{Q}) \to C_{d-1}(L;\mathbb{Q})}{\operatorname{Im} \partial : C_{d+1}(L;\mathbb{Q}) \to C_d(L;\mathbb{Q})}$$

"cohomological degree"

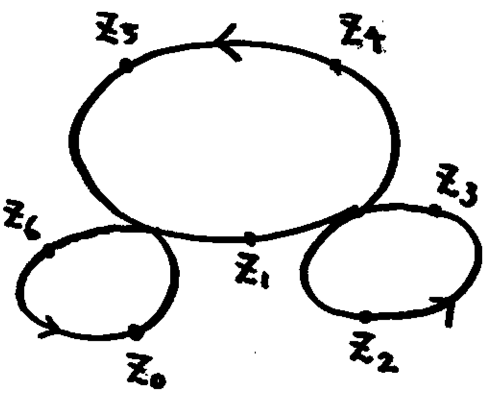
$$C^{n-d}(L;\mathbb{Q}) := C_d(L;\mathbb{Q}), \quad n = \dim L$$

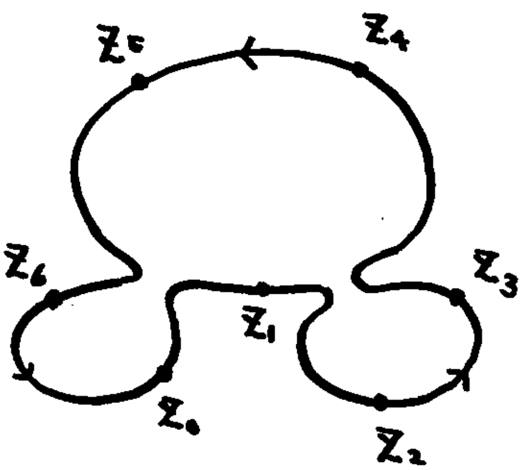
(Σ, j) a connected, genus 0, bordered Riemann surface with $\underbrace{\text{nodal singularities}}_{xy=0}$



$k \geqq 0$

$z_0, \dots, z_k$ smooth points of $\partial\Sigma$ ordered by j

$\partial\Sigma =$ 

$\Longrightarrow$ 

smoothing

J an almost complex structure on M

$u : \Sigma \to M$, $u(\partial\Sigma) \subset L$, stable

iff $\quad J \circ du = du \circ j$

&

$$\# \mathrm{Aut}(\Sigma, z_0, \dots, z_k, u) < \infty$$

$$\mathrm{Aut}(\Sigma, z_0, \dots, z_k, u) := \{ \varphi : \Sigma \to \Sigma \text{ biholomorphic s.t. } \varphi(z_i) = z_i \text{ \& } u \circ \varphi = u \}$$

$\beta \in H_2(M, L)$

$$\mathcal{M}_{k+1}(\beta) := \left\{ (\Sigma, z_0, \ldots, z_k, u) \text{ stable maps} \quad \text{s.t. } u_*[\Sigma] = \beta \right\} \Big/ \sim$$

$$(\Sigma, z_0, \ldots, z_k, u) \sim (\Sigma', z_0', \ldots, z_k', u')$$

iff $\exists\, \varphi : \Sigma \to \Sigma'$ biholomorphic

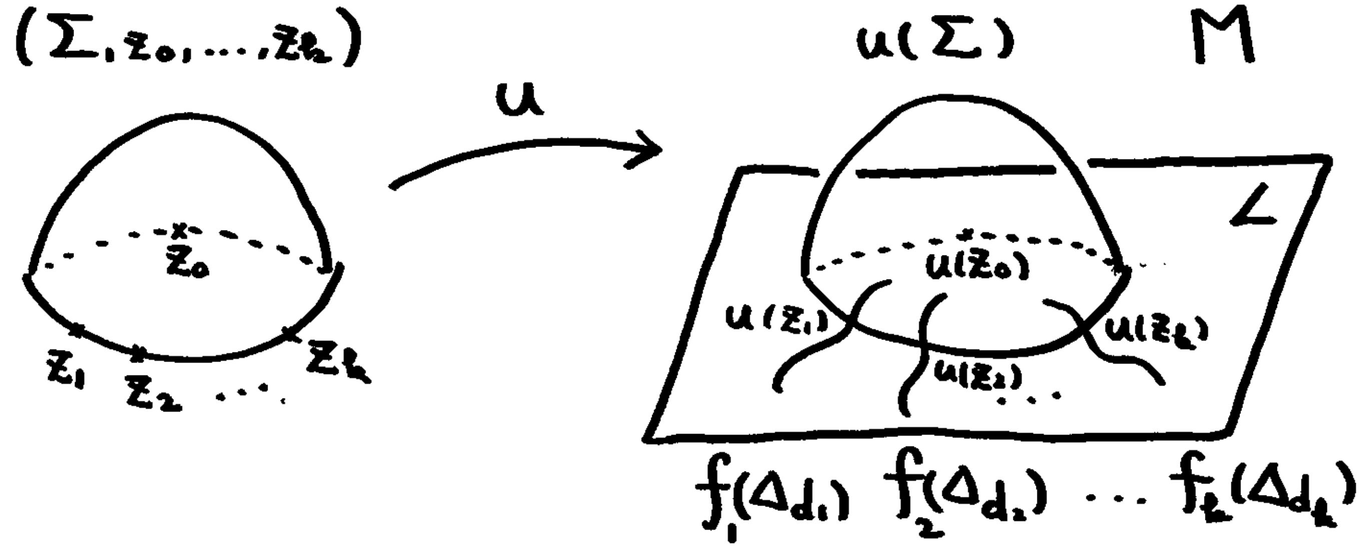
s.t. $\varphi(z_i) = z_i'$ & $u' \circ \varphi = u$

$$f_i : \Delta_{d_i} \to L, \quad 1 \leqq i \leqq k$$

$$\mathcal{M}_{k+1}(\beta, f_1, \ldots, f_k)$$

$$:= \left\{ [\Sigma, z_0, \ldots, z_k, u] \in \mathcal{M}_{k+1}(\beta) \right.$$

$$\left. \text{s.t. } u(z_i) \in f_i(\Delta_{d_i}),\ 1 \leqq i \leqq k \right\}$$



$$\begin{array}{ccc} ev : \mathcal{M}_{k+1}(\beta, f_1, \dots, f_k) & \longrightarrow & L \\ \Psi & & \Psi \\ [\Sigma, z_0, \dots, z_k, u] & \longmapsto & u(z_0) \end{array}$$

$$\text{"} \in C_*(L; \mathbb{Q}) \text{"}$$

$$* = n + \underbrace{\mu(\beta)}_{\text{the Maslov number of } \beta} + k - 2 + d_1 + \cdots + d_k - kn$$

$k \geqq 0$

$$m_{k.\beta} : C^{D_1}(L;\mathbb{Q}) \otimes \cdots \otimes C^{D_k}(L;\mathbb{Q}) \to C^{-\mu(\beta)+D_1+\cdots+D_k+2-k}(L;\mathbb{Q})$$

linear maps

$(k.\beta) = (1.0)$

$$m_{1.0}(f) := \partial f$$

$(k.\beta) \neq (1.0)$

$$m_{k.\beta}(f_1,\dots,f_k) := \pm \text{"}ev : \mathcal{M}_{k+1}(\beta, f_1,\dots,f_k) \to L\text{"}$$

Novikov ring

$$\Lambda := \left\{ \sum_i a_i e^{\mu_i} T^{\lambda_i} \text{ formal power series s.t. } a_i \in \mathbb{Q},\ \mu_i \in \mathbb{Z},\ \lambda_i \in \mathbb{R},\ \lim_{i \to \infty} \lambda_i = \infty \right\}$$

$\deg e = 1$, $\deg T = 0$

$$m_k : C^{D_1}(L;\Lambda) \otimes \cdots \otimes C^{D_k}(L;\Lambda) \to C^{D_1+\cdots+D_k+2-k}(L;\Lambda)$$

Λ-linear maps

$$m_k := \sum_{\beta \in H_2(M,L)} e^{\mu(\beta)} T^{\omega(\beta)} m_{k,\beta}$$

Fukaya-Oh-Ohta-Ono

$$\left(C^*(L;\Lambda), \{m_k\}_{k=0}^{\infty} \right) \text{ an } A_\infty\text{-algebra}$$

$k \geqq 0$

$$\sum_{\substack{k_1+k_2=k+1 \\ 1 \leqq i \leqq k_1}} \pm m_{k_1}(f_1, \ldots, f_{i-1}, m_{k_2}(f_i, \ldots, f_{i+k_2-1}), f_{i+k_2}, \ldots, f_k)$$

$$= 0$$

$k=0 \qquad m_1(m_0(1)) = 0$

$k=1 \qquad m_2(m_0(1), f) \pm m_2(f, m_0(1))$

$\qquad\qquad \pm m_1(m_1(f)) = 0$

$k=2 \qquad m_3(m_0(1), f_1, f_2) \pm m_3(f_1, m_0(1), f_2) \pm m_3(f_1, f_2, m_0(1))$

$\qquad\qquad \pm m_2(m_1(f_1), f_2) \pm m_2(f_1, m_2(f_2))$

$\qquad\qquad \pm m_1(m_2(f_1, f_2)) = 0$

$\vdots$

In general, $m_0(1) \neq 0$

&

$m_1 m_1 \neq 0$

$$b \in C^1(L:\Lambda)$$

$$m_k^b(f_1,\dots,f_k) := \sum_{l_0,\dots,l_k} m_{k+l_0+\dots+l_k}(\overbrace{b,\dots,b}^{l_0},f_1,\overbrace{b,\dots,b}^{l_1},f_2,b,\dots,b,f_k,\underbrace{b,\dots,b}_{l_k})$$

$$\Rightarrow (C^*(L:\Lambda),\{m_k^b\}_{k=0}^{\infty}) \text{ an } A_\infty\text{-algebra}$$

b a bounding cochain

iff $m_0^b(1) = 0$

$\left(\Leftrightarrow m_0(1) + m_1(b) + m_2(b,b) + \cdots = 0 \right)$

b a bounding cochain $\Rightarrow m_1^b m_1^b = 0$

Floer homology of (L, b)

$$HF(L,b) := \frac{\operatorname{Ker} m_1^b}{\operatorname{Im} m_1^b}$$