On Mathematical Problems of Quantum Field Theory

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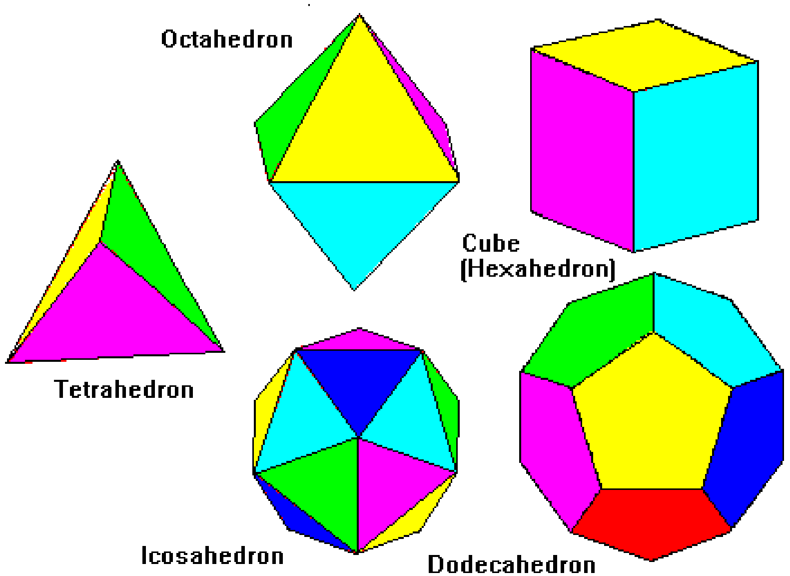
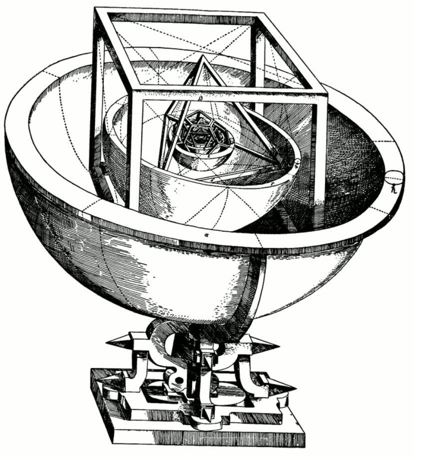
Antiquity

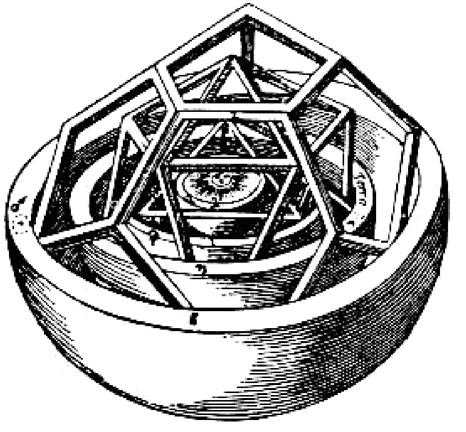
- Plato: a model of the universe, 5 elements, Platonic solids,
- Developed the concept of a mathematical vision of the universe, the “world of ideal substances”,



Renaissance

- Kepler: laws of planetary motion
- Observed that ratios of orbits of all known planets are numerically close to ratios of Platonic solids inscribed into spheres.
- Thus, he "explained" the known universe in terms of mathematics revered by classical Greeks.



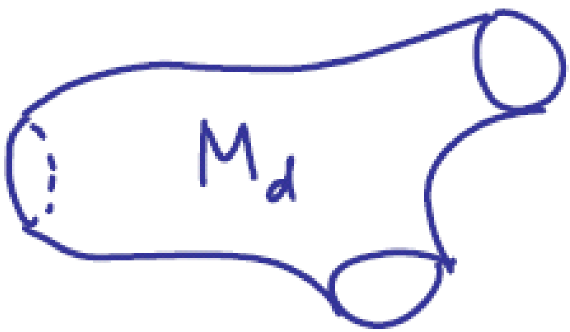


Modern Times: physics---mathematics

- Classical Mechanics (Newton, Lagrange, Hamilton,…) ---Differential Equations and Symplectic Geometry
- Electro-Magnetism (Maxwell)---Differential Geometry
- Special Relativity (Einstein-Lorentz-Poincare)--- Representation theory of non-compact Lie groups
- General Relativity (Einstein)---Riemannian Geometry
- Quantum Mechanics (Bohr,Heisenberg,Plank,… Hilbert,Von Neumann,…)---Operators in Hilbert Spaces, Operator Algebras, Functional Analysis,…..
- Quantum Field Theory (Feynman, Dirac, …..)—All of the above and new geometric and algebraic structures.

Quantum Field Theory

① Any QFT does the following:



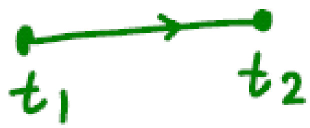
$\rightsquigarrow Z(M) \in H(\partial M)$

$\partial M_d = (d-1)$ - dimensional

- $\mathcal{H}(\partial M)$ = the space of state
- $Z(M)$ = the partion function, propagator, amplitude

Example: $d=1$, Quantum mechanics

M



$$\rightsquigarrow \sum_a x^a \otimes e^{i\frac{H}{\hbar}(t_2-t_1)}\, x_a$$

$$\cap$$

∂M

$$\rightsquigarrow \mathcal{H}(t_1) \otimes \mathcal{H}(t_2) = \mathcal{H}^* \otimes \mathcal{H}$$

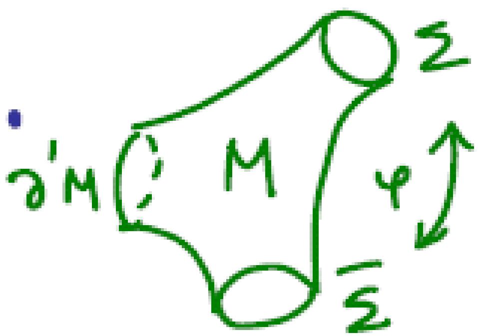
$$\mathcal{H}^* \otimes \mathcal{H} \simeq \mathrm{End}(\mathcal{H}), \quad \sum_a x^a \otimes e^{i\frac{H}{\hbar}(t_2-t_1)}\, x_a \leftrightarrow$$

$$\leftrightarrow \; e^{i\frac{H}{\hbar}(t_2-t_1)} : \mathcal{H} \circlearrowleft$$

② Locality:

- $H(N_{d-1} \sqcup N'_{d-1}) = H(N_{d-1}) \otimes H(N'_{d-1})$

- $Z(M_1 \sqcup M_2) = Z(M_1) \otimes Z(M_2) \in$
  $\in H(\partial M_1) \otimes H(\partial M_2)$

- 

$\partial M = \partial' M \sqcup \Sigma \sqcup \overline{\Sigma}$

$M_{(\Sigma,\varphi)}$ = result of gluing

$$\sum_{a \in H(\Sigma)} Z(M)^{a}_{a} = Z(M_{(\Sigma,\varphi)})$$

③ Classical limit: parameter $\hbar$

(i) "Physically", the limit when

$$\langle \mathcal{O} \rangle \gg \hbar \quad , \quad \mathcal{O} = \text{"generic observable"}$$

(ii) "Mathematically", $\hbar \to 0$.

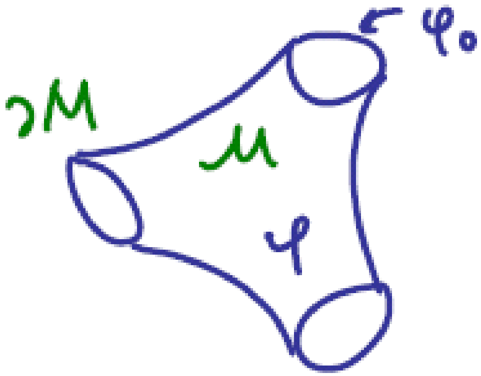
In this limit expectation values of can be expressed in terms of classical fields (evolving as solutions to non-linear PDE's).

④ <u>Symmetry</u>:

$$T(g)\, Z(M) = Z(M^g) \in H(\partial M^g)$$

$$T(g) : H(\partial M) \to H(\partial M^g)$$

Path integral:

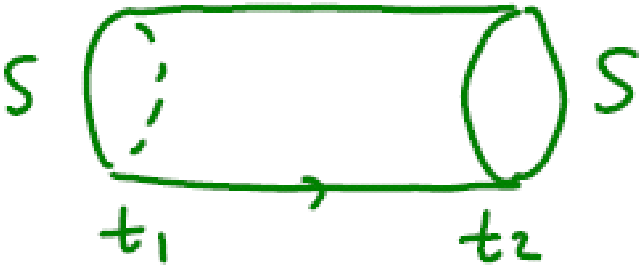


$$Z(M|\varphi_0)=\int_{\varphi|_{\partial M}=\varphi_0} e^{\frac{i\,S(\varphi)}{\hbar}}\,\mathcal{D}\varphi \;=\; ?$$

$$? = \begin{cases} \displaystyle\sum_{\varphi_c} \frac{1}{\sqrt{S''(\varphi_c)}}\, e^{\frac{i\,S(\varphi_c)}{\hbar}} \sum_{\text{F. diagr}} Z_\Gamma(\varphi_c), & \text{semi-classical} \quad \begin{cases} S'(\varphi_c)=0 \\ \varphi_c|_{\partial M}=\varphi_0 \end{cases} \\[2ex] \text{the limit of finite-dim. integrals} & \text{(constructive field theory)} \end{cases}$$

Operators :

$\mathcal{M} = [t_2, t_1] \times S$



$\mathcal{H}(\partial M) = \mathcal{H}(\bar{S}) \otimes \mathcal{H}(S) \simeq$

$\simeq$ operators in $\mathcal{H}(S)$

$Z(M) : \mathcal{H}(S) \to \mathcal{H}(S)$ operator

$\mathcal{H}(S) = \begin{cases} \text{functionals on fields on } S, \text{ semi-classically} \\ \text{Fock space, when there are (massive) particles} \end{cases}$

Realistic Models

- Main structural features:

--- Gauge symmetry

--- Very specific field content (from experiment)

--- 3+1 dimensional

- What is known:

---Renormalizable quantum field theories (in perturbation theory)

---Asymptotically free

---First few orders of the perturbation theory agree with experiments !!!

- We want to know (the Y-M millennium problem):

  --- How to construct this quantum field theory non-perturbatively ? (math)

  --- Why there is a confinement in gauge theories, i.e. why we do not see quarks? (math)

- Higgs, supersymmetry,….(physics)

Physics or Mathematics?

- These problems are physically guided mathematical problems: there are first principles, experiments (physical guidance), from which the details of the theory should follow by means of mathematical tools.
- These problems are also formidably complicated, both to formulate (in a physically and mathematically meaningful way), and to solve, thus increasing interaction between physics and mathematics.

Non-perturbative QFT’s

- Conformal Field Theories in 2D
- Chiral Gross-Neveu, Principal Chiral Field, …., Sine-Gordon, ….
- Topological Quantum Field Theories. Chern-Simons theory.

Conformal Field Theory

- Conformal invariance in 2D:

$$z \mapsto z + \varepsilon(z), \quad \bar{z} \mapsto \bar{z} + \bar{\varepsilon}(\bar{z})$$

∞-dimensional (Lie algebra) space of infinitesimal transformations

- Field $\Phi(z, \bar{z})$ has conformal dimension $(\Delta, \bar{\Delta})$ if

$$\Phi(z,\bar{z}) \mapsto \Phi(z,\bar{z}) + \varepsilon(z)\frac{\partial \Phi}{\partial z} + \bar{\varepsilon}(\bar{z})\frac{\partial \Phi}{\partial \bar{z}}$$
$$+ \Delta\, \varepsilon'(z)\,\Phi(z,\bar{z}) + \bar{\Delta}\,\bar{\varepsilon}(\bar{z})\,\Phi(z,\bar{z})$$

- Possible values of $(\Delta, \bar{\Delta})$ are determined by the Lie algebra of conformal transformations.

(Belavin, Polyakov, Zamolodchikov, $\mathbb{R}^2$)

(Tsuchiya, Ueno, Yamada )

Principal Chiral Field

Classical action: field $g(x) \in SU(N)$,

$x \in$ flat space-time 2D

$$S[g] = \sum_{i=1}^{2} \int_{\mathcal{M}} \mathrm{tr}\left(g(x)^{-1} \partial_i g(x)\, g^{-1}(x)\, \partial_i g(x) \right) d^2x$$

Particle spectrum: • $m_\ell = m \sin\left(\frac{\pi \ell}{N} \right)$

(Polyakov, Wiegmann) $\ell = 1, .., N-1$

- (ℓ) transforms according to $V^*_{\omega_\ell} \otimes V_{\omega_\ell}$
- $(N-\ell)$ is an antiparticle of (ℓ)
- (ℓ) is a bound state of ℓ (1)-particles

Complete set of local correlation functions is constructeg (Smirnov, 92)

$$\langle g_{i_1 j_1}(t_1, x_1) \cdots g_{i_n j_n}(t_n, x_n) \rangle$$

- Asymtotical freedom
- dynamical mass generation
- Representation theory of certain ∞-dimensional algebras.

Along these lines fascinating new results (Jimbo, Miwa, Smirnov, ...)

Topological Quantum Field Theory

Does not depend on Riemannian structure on $\mathcal{M}$ (no distances in the theory).

- Chern-Simons theory (Witten)
  A path integral;
  Wess-Zumino-Witten conformal field theory at the boundary
- There is a finite dimensional QFT ($\mathcal{H}(\partial M)$ is finite dimensional) constructed algebraically (Turaev, R)
- Should have the same semi-classical limit (?)

Some perspectives

- Integrability in 4D SUSY YM and $AdS_5 \times S_5$
- Connes-Kreimer infinite-dimensional Lie algebras and the renormalization.
- Conformal field theories in 2D with boundary. (Belavin-Polyakov-Zamolodchikov;Tsuchia-Yamada-Ueno;…)
- Quantum field theory on manifolds with boundaries via semi-classical expansion of the path integral.
- Semiclassical limit in the Chern-Simons (Witten, Axelrod-Singer, Kontsevich,…). Comparison with combinatorial formulae (Turev-R, …)
- Classification of topological field theories based on BV formalism (Schwarz, Kontsevich,…)
- Correlation functions in integrable models (Jimbo, Miwa, Smirnov,…).
- Topological string theory as the large N limit of Chern-Simons (Witten, Vafa, Gross-Taylor,…).