Geometric Structures over Space and Their Applications to Physics

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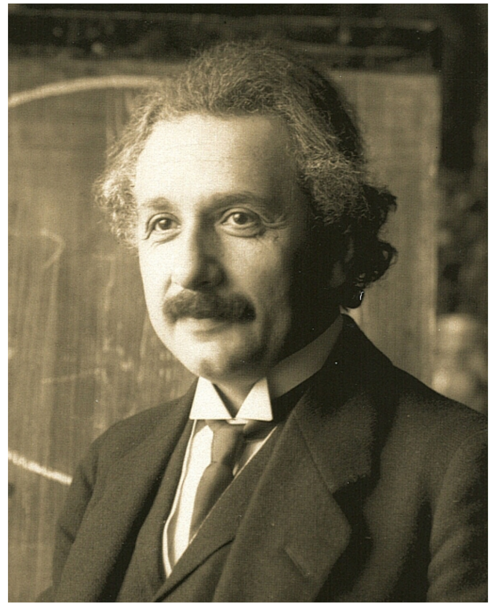
Geometry is perhaps one of the oldest branch of mathematics and is closest to fundamental physics. Geometry and number theory are two subjects that form the core of modern mathematics.

By geometry, I am referring to the “practical geometry” as defined by Einstein. I quote some remarks of his given at the Prussian Academy of Sciences in 1921:



Albert Einstein

“(Practical) geometry, owes its existence to the need which was felt of learning something about the behavior of real objects.”

“We may in fact regard it as the most ancient branch of physics. Its confirmations rest essentially on induction from experience, but not on logical inferences.”

“We will call this completed geometry ‘practical geometry,’ and shall distinguish it in what follows from ‘purely axiomatic geometry.’ ”

There are many ways that geometry provides both the language and the intuition for the physical world. Let us focus what we call geometric structures on space (time) that have been the major topics in modern geometry.

I. Geometric structures built by local patching based on local symmetries

In the 19th century, Sophus Lie started the investigation of the continuous group of symmetries. Klein, in his famous Erlangen program announced that one should study geometry according to the symmetries of the geometric structures.

Sophus Lie Felix Klein

Some examples can be found in classical (extrinsic) geometries where one studies differential invariants of the submanifolds that are invariant under the projective group, the affine group or the Möbius group acting on the ambient space.

The concept of distance in these geometries is no longer important. When one attempts to go beyond subspaces of Euclidean space and extend the above concepts to subspaces of a manifold, one needs to construct geometric structures over manifolds where coordinate transformations preserve the above symmetries.

An important example is the modern concept of Riemann surface. It was first (rigorously) defined by Weyl to be a space which is covered by coordinate charts where the coordinate transformation are holomorphic mappings of one variable.

Hermann Weyl

It gave rise to the concept of geometric structures defined by those spaces that can be covered by coordinate charts where the coordinate transformations belong to some special group such as the (pseudo-)group of holomorphic transformations, the group of projective transformations, the group of affine transformations or the group of conformal transformations.

An important question which has not been solved up to now is to find the topological criterion for the existence of such structures. In general, these are deep questions. Consider even the simple question

Which manifolds admit flat affine structures?

We do not know whether the Euler number of such manifolds should be zero or not. But we do know that spheres cannot be covered by coordinate charts such that the transition functions are linear.

II. Geometric structures in terms of connections

It is difficult to follow Klein's program for general spaces as they may not admit any group of symmetries. Cartan studied the concept of connections on the bundle of frames over a manifold.

Elie Cartan

The enlarged internal space then has a large continuous group of symmetries (while the space itself may not support any symmetries whatsoever). This allowed Cartan to carry out the Klein program for general manifolds.

The space of frames is called the principle bundle nowadays. The connection on such bundles allows one to define a covariant derivative which is compatible with the frame. The frames may be special and can be identified with special group of symmetries such as orthogonal group, unitary group, *etc*.

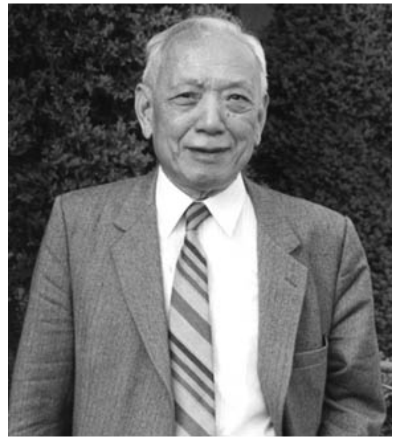
Parallel transportations along close paths carry frames of special type back to frames of special type. In this case, we say that the holonomy group is this group of special type. Indeed, the holonomy group is a very powerful symmetry group of the geometry that dictates the geometry of the space.

It should be noted that much of the development of parallel transportation (due to Levi-Civita in 1917 and Schouten in 1918) and connections on space of frames were motivated by the excitement of Einstein's theory of general relativity. However, these authors were mostly interested in connections on the tangent bundle.

Around 1918, Weyl started to study Abelian gauge theories to unify gravity with Maxwell's equations. Weyl first used noncompact gauge group. In 1927, Fock and London pointed out that in quantum mechanics, Weyl's connection should be made purely imaginary. Weyl's theory (as Weyl noticed himself in 1929) then became a gauge theory with the structure group of a circle. The connection now preserves lengths, overcoming an objection of Einstein.

The theory of fiber bundle was developed in more detail soon afterwards by Cartan and his students: Ehresmann, Chern and others.

The topologists Whitney and Pontryagin made fundamental contributions towards the basic invariants of fiber bundle: their characteristic classes.



Shiing-Shen Chern

In the works of Cartan and Chern, they were very much interested in the equivalence problem in geometric structures: basically one likes to find all calculable invariants of the geometric structures so that one can determine the complete local structure of the geometry.

A good example is to determine the complete local invariants of a real (pseudoconvex) hypersurface in complex Euclidean space that are invariant under the group of biholomorphic transformations of the complex Euclidean space. This is a problem that Cartan, Tanaka, Chern-Moser made fundamental contributions in.

If the frames are the frames of the tangent space of the manifold, the connection has another feature: the torsion tensor of the connection, defined by

$$D_X Y - D_Y X - [X, Y]$$

and the connection is said to be symmetric if the torsion tensor vanishes.

The torsion tensor has not been understood well. In many occasions, it can be considered to be the integrability condition for geometric structures to exist.

Geometric structures can be constructed in several steps. The basic goal is to find a computable criterion to check whether a manifold admits a geometric structure.

A good example is to construct complex structures over a given topological manifold. (Those manifolds that can be covered by coordinate charts in complex Euclidean spaces where the coordinate transformations are holomorphic.)

If a complex structure exists, then there is an algebraic structure on the tangent space of the manifold. It is called an almost complex structure where an endomorphism J acts on the tangent space with $J^2 = -$identity. Its eigenvalues are i or $-i$.

There is a way to determine whether a manifold admits a J structure or not. The conditions can be written in terms of computable invariants such as characteristic classes. So the existence theory is quite satisfactory.

However, an important problem is to determine whether the J structure can be constructed from a complex structure of the manifold or not. In other words, we need to construct locally defined coordinate functions that solves the $\bar{\partial}$ equations.

This was accomplished by the Newlander-Nirenberg theorem where the torsion-free condition is the condition for an almost complex structure to be integrable. However, the problem to find a topological criterion for a smooth manifold to support an almost complex structure which is torsion-free is still mysterious.

Kunihiko Kodaira

For complex dimension two, the deep works of Kodaira, based on the Atiyah-Singer index theorem, shows that there are many manifolds that supports some almost complex structure, but cannot admit any integrable complex structure.

A definite answer to the question whether an almost complex manifold can be deformed to admit an integrable complex structure will go a long way to understand the topology of four dimensional manifolds.

On the other hand, the question for complex dimensions greater than two can be easier. I conjecture that almost complex manifold with complex dimensions greater than two must admit an integrable complex structure.

The above mentioned holonomy group is a very important algebraic entity associated to the connection. The holonomy groups of the Levi-Civita connection were classified by the works of Berger-Simons. The possible holonomy groups are

1. Unitary group, which implies the manifolds admit a Kähler structure.

2. Special unitary group, which implies the manifold is a Calabi-Yau manifold

3. Intersection of the unitary group with the symplectic group, which implies the manifold is hyperkähler.

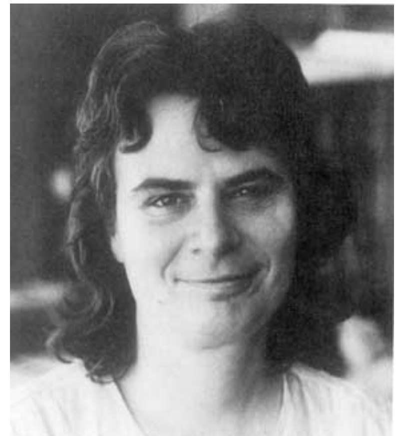
4. $Sp(m)Sp(1)$, $m \geq 2$, the manifold is quaternionic Kähler.

5. Exceptional holonomy groups G_2 and $Spin(7)$.

The Ricci curvature of these manifolds in class 2, 3, and 5 are all trivial. Hence it satisfies the vacua solution of the Einstein equations.

In the past 25 years, a great deal of efforts were devoted to understanding manifolds with special holonomy group partly because they are related to string or M-theory. The Calabi conjecture gives a satisfactory answer for $SU(n)$ and $Sp(n)$ holonomy. Joyce constructed manifolds with G_2 and $Spin(7)$ holonomy. But Joyce's constructions are not complete enough to give a full parametrization of the geometric structures.

We are also interested in those connections on a vector bundle with special structures. For four-manifolds, there are connections whose curvature two-form satisfies a duality condition. They are anti-self-dual connections on a Kähler surface and can be generalized to Hermitian Yang-Mills connections on a holomorphic bundle of a higher dimensional Kähler manifold. These bundles play an important role in heterotic string theory.



Simon Donaldson   Karen Uhlenbeck

Donaldson-Uhlenbeck-Yau proved their existence to be equivalent to an algebraic condition that the bundle is stable in the sense of geometric invariant theory. Generalization was made by Simpson to include a Higgs field. The Hermitian Yang-Mills-Higgs theory can be applied to construct geometric structures over Kähler manifolds.

III. Geometric structures obtained by the reduction of the geometric structures defined by special holonomy group

There are geometric structures obtained by taking a special ansatz to reduce the full non-linear equation to simpler equations with less variables. The most common practice is to apply group actions and study the orbit spaces. The development of moment map and its symplectic reduction is a powerful example.

In this way, equations of metrics with special holonomy can be reduced to structures of special interest. For example, the concept of Sasaki-Einstein metric is obtained by a reduction on a Calabi-Yau cone. Such metrics appear recently in the string theory context of AdS/CFT duality.

IV. Structures defined by natural equations

The most important equations in geometry are related to Einstein equations. Both the Lorentzian and the Riemannian version have been important in the history of geometry.

In the Riemannian category, finding Einstein metrics with zero or nonzero cosmological constant is one of the most important question in geometry.

Another set of interesting questions come from the study of dynamics of metrics. The most spectacular equations is the Einstein equation for spacetime. In order for a three dimensional manifold to be a spacelike hypersurface in a vacuum spacetime, there are certain constraints it must satisfy. There is a metric tensor g_{ij} with scalar curvature R and a symmetric tensor p_{ij} on the manifold so that

$$R - \|p\|^2 + (\operatorname{tr} p)^2 = 0$$

$$p^i{}_{j,i} - (p^i{}_i)_{,j} = 0$$

Under mild conditions, Schoen and I classified the topological structures of such three dimensional manifold. They can be described by connecting many copies of the tubes - the product of a two dimensional sphere with circle - with those three manifolds that are covered by the three spheres. We proved that if a region within a fixed region of space is filled with matter such that the matter density exceeds a certain value, then a black hole must form in this region.

The space of structures that satisfy the constraint equations is infinite dimensional. But it is invariant under the flow defined by the Einstein equation. Hence, the parametrization of this infinite dimensional space is important. Some work was done by R. Bartnik and Schoen and his students.

Richard Hamilton

Another important dynamical system related to the Einstein metric is given by Hamilton's Ricci flow. During his investigation of the singularities of the flow, he needed to study those spaces that evolve like a soliton. Such a structure generalizes the concept of the Einstein metric.

V. Geometric structures related to string theory

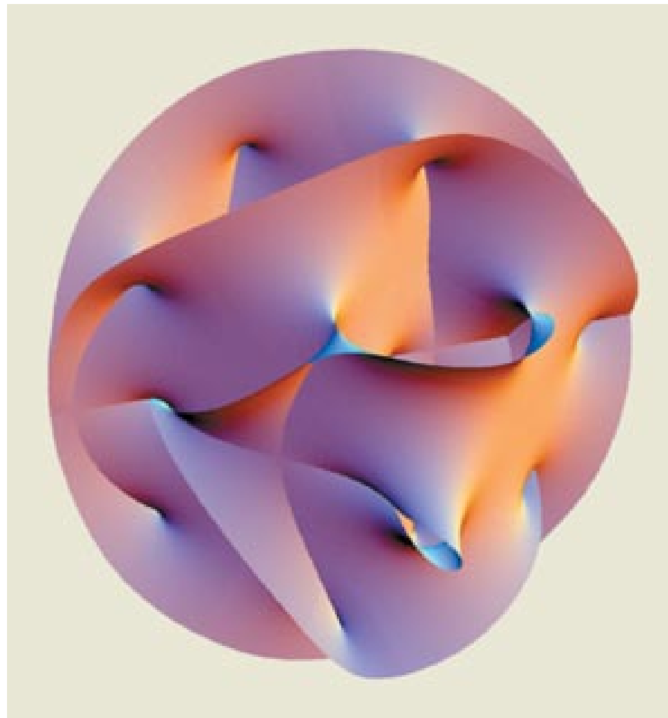
Supersymmetry has been a powerful concept in modern geometry.

In the past 25 years, much development of spaces with special holonomy groups came from the works of physicists. It is fair to say that without the input of string theorists, the development of many aspects of this subject would have taken a much longer time. And without the contributions of mathematicians, string theorists would have lost their confidence in the consistency of their theory, at least at the level of abstract theory.

For geometric structures, string theory can provide many fresh new ideas. However, physical intuitions do not give a mathematical proof of the existence of the geometric structures. In most cases, it takes a great deal of analysis to prove the existence of structures. (Sometimes, one can prove such structures cannot exist and physicists would call them no-go theorems. Finding an existence or non-existence theorem is a good test for theories proposed by physicists.)

It is a nontrivial task to determine which geometric structure can exist on a manifold. In fact, I recall that about thirty-eight to forty years ago, most mathematicians did not believe that non-trivial Kähler manifolds with vanishing Ricci curvature exists.

Many unexpected applications on Calabi-Yau manifold were discovered by string theorists. The most important one was the discovery of the concept of various dualities relating different string models. Since many of the models were built on Calabi-Yau manifolds, they can be checked by calculations on such manifolds.



A Quintic Calabi-Yau (by A. Hanson)

Vafa *et al.* has developed the concept of quantum cohomology for Calabi-Yau manifolds and their properties such as associativity were studied by WDVV (Witten, Dijkgraaf, E. Verlinde and H. Verlinde).

The discovery of mirror symmetry by Greene-Plesser, and Candelas *et al.* have opened up the eyes of most algebraic geometers who are interested in enumerative geometry, a subject where algebraic geometers calculate the number of curves with fixed degree and genus.

An important consequence of the works on mirror symmetry is the effective calculation of such numbers for genus zero in a Calabi-Yau manifold. In the old days, mathematicians did not have any idea how to find the formula for the number of curves. Ideas from conformal field theory, via mirror symmetry, led the way to find such a formula. While the ideas from string theory was not good enough to provide a proof, a rigorous proof was found independently by Givental and Lian-Liu-Yau. It solved an old problem in enumerative geometry. It can also be considered as a nontrivial consistency check of ideas of string theory.

However, the problem of counting algebraic curves of higher genus remains one of the most challenging problem in both string theory and mathematics. The spectacular work of BCOV (Bershadsky, Cecotti, Ooguri and Vafa) was a major step toward accomplishing this goal. But many questions remain before we can find a robust formula for higher genus.

There are other dualities that appeared in string theory. Each one of them has given rise to new mathematical insights into geometry.

About ten years ago, Strominger-Yau-Zaslow, proposed a new geometric way to understand mirror symmetry, by viewing it as a duality along special Lagrangian torus fibrations. At around the same time, Kontsevich proposed his homological mirror conjecture which predicts that the Fukaya category is isomorphic to the derived category of the mirror manifolds.

All these approaches opened up many paths to analyze the geometry of Calabi-Yau manifolds, and many conjectures in mathematics were made and proved based on these two approaches.

A rather interesting point realized is that the Calabi-Yau $SU(3)$ structure can be calculable in terms of several data of the mirror. Another remarkable statement is that the number and the area of the holomorphic disks will contribute to build the metric of the mirror manifolds.

Hence for building a geometric structure, not only should we look for metrics with certain holonomy group, but we should also study other objects which physicists called branes. I think gradually, a new concept of geometric structure will emerge out of combining information of metrics with special structures, submanifolds of special structures, and bundles with special structures. It will take some time to explore all these geometric structures combine.

The concept of supersymmetry has played an important role. Let me give an example.

Many years ago, Strominger was interested to build supersymmetric heterotic string models with H-flux. One needs to analyze the hermitian metric on the manifold and also the metrics on holomorphic bundles. Both of them should satisfy supersymmetric conditions. But they are also linked by an anomaly equation. They can be written down as the following four equations:

$$d(\parallel \Omega \parallel_\omega \omega^2) = 0 \ ,$$

$$F^{2,0} = F^{0,2} = 0 \ ,$$

$$F \wedge \omega^2 = 0 \ ,$$

$$\sqrt{-1}\,\partial\bar{\partial}\omega = \alpha'(\mathrm{tr}R \wedge R - \mathrm{tr}F \wedge F) \ .$$

These are a natural set of equations. But finding solutions to this set of equations turned out to be difficult. Only recently, Jun Li-Yau and Fu-Yau were able to prove the existence of solutions to such equations.

While it may still be a long way to go to find a geometry that can incorporate both quantum theory and general relativity, we have already come across many interesting questions in geometry along the way. The idea of studying coupled sets of objects together was not so popular in geometry. We hope understanding their structures will lead to new insights.

The concept of spacetime has been evolving since the ancient days. The challenge of building a meaningful quantum geometry certainly will take efforts of all theoretical scientists. There is no doubt that the institute here will be one of the key foundation for such enterprise.